# Asymptotic Normality of Occupation Time of Singularly Perturbed Diffusions

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2013.7.6-10, Ermei,Sichung

# Introduction and summary of results 1



Let $L^\epsilon(t, x)$ be a two-time scale diffusion generator on the unit circle $S^1$ of the form

$$L^\epsilon(t, x) = 1/\epsilon \cdot L_1(t, x) + L_2(t, x)$$

where $L_i(t, x) = b_i(t, x)\partial_x + 1/2 \cdot a_i(t, x)\partial_{xx}$. $b_i(t, x), a_i(t, x) > 0$ are smooth functions on $S^1$. Let $X_t^\epsilon$ satisfy the following stochastic differential equation :

$$\begin{aligned} dX_t^\epsilon &= (1/\epsilon \cdot b_1(t, X_t^\epsilon) + b_2(t, X_t^\epsilon))dt \\ &\quad + \sqrt{1/\epsilon \cdot a_1(t, X_t^\epsilon) + a_2(t, X_t^\epsilon)}dW_t \\ X_0^\epsilon &= \delta_x. \end{aligned}$$

# Introduction and summary of results 2



There exists functions $p_i(t,y)$ and $q_{i,x}(t,y)$ such that

- $$sup_{t\le T, y\le 1} \quad |p^\epsilon(t,y) - (\Sigma_0^n \epsilon^i p_i(t,y) + \Sigma_0^n \epsilon^i q_{i,x}(t/\epsilon, y))| = O(\epsilon^{n+1}).$$

- $p^\epsilon(t,y)$ is the distribution of $X_t^\epsilon$, $\Sigma_0^n \epsilon^i p_i(t,y)$ is called the regular part $\Sigma_0^n \epsilon^i q_{i,x}(t/\epsilon,y)$ is the singular part of $p^\epsilon(t,y)$ and $|q_{i,x}(t,y)| \le Ke^{-\gamma t}$ as $t \to \infty$ uniformly over $x, y$ and $i \ge 1$.
- $P_n^\epsilon(t,y) := \Sigma_0^n \epsilon^i p_i(t,y) + \Sigma_0^n \epsilon^i q_{i,x}(t/\epsilon, y)$.
- Note that the regular part $\Sigma_0^n \epsilon^i p_i(t,y)$ does not depend on $x$ and $p_0(t,y) = p(t,y)$ is the quasi-stationary distribution of $L_1(t,x)$.
- $b_i(t,x)$ and $a_i(t,x) \in C^{n+1,2(n+1)}[0,T] \times S^1$.

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# Introduction and summary of results 3



For a bounded function $f(x)$ on $S^1$, an unscaled function of the occupation time of $X_t^\epsilon$ is defined as follows.

- $$Z^\epsilon(t,f) = \int_0^t \left( f(X_s^\epsilon) - \int_0^1 f(y)p(s,y)dy \right) ds,$$
- A scaled function is defined as :

$$n^\epsilon(t,f) = 1/\sqrt{\epsilon} \cdot Z^\epsilon(t,f).$$

- We shall establish a weak law of large numbers :

$$lim_{\epsilon \to 0} EZ^\epsilon(t,f)^2 = 0, \text{ and thus} Z^\epsilon(t,f) \to 0 \text{ in probability},$$

- and asymptotic normality :

$$n^\epsilon(t,f) \to n(t,f) \text{ in C[0,T] where } n(t,f) \text{ is Gaussian.}$$

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The covariance function of $n(\cdot,f)$, independently of the initial point, is as follows.

- $$En(t,f)^2 = 2\int\int\int_0^t\int_0^\infty f(x)f(y)p(r,x)q_{0,r,x}(u,y)dudrdxd$$
- Moreover, $n(\cdot,f)$ has independent increment and $E(n(t,f)\cdot n(s,f)) = En^2(t\min s, f)$.

# Asymptotic expansion of $X_t^\epsilon$ 5



Let $L^\epsilon = 1/\epsilon \cdot L_1(t,x) + L_2(t,x)$ and $p^\epsilon(t,y)$ be the density of $X_t^\epsilon$ with initial distribution $\delta_x$.

- $p^\epsilon(t,y)$ satisfies the forward equation of $L^\epsilon$, i.e.,

$$\partial_t p^\epsilon(t,y) = L^{\epsilon,*}(t,y)p^\epsilon(t,y), p^\epsilon(0,y) = \delta_x(y).$$

- $L^{\epsilon,*}(t,y)\cdot = -\partial_y((1/\epsilon b_1(t,y) + b_2(t,y))\cdot) + 1/2\partial_{yy}((1/\epsilon a_1(t,y) + a_2(t,y))\cdot)$ is the adjoint operator of $L^\epsilon$.

# Outline

# Regular Part 6



For the regular part, we have $(\partial_t - L^{\epsilon,*})(\Sigma_0^n \epsilon^i p_i(t,y)) = 0$. Collecting terms, we have

$$\begin{aligned}
\epsilon^{-1}, &\quad L_1^* p_0 = 0, \\
\epsilon^{0}, &\quad L_1^* p_1 = \partial_t p_0 - L_2^* p_0 \\
\epsilon^{1}, &\quad L_1^* p_2 = \partial_t p_1 - L_2^* p_1 \\
&\quad .... \\
\epsilon^{n}, &\quad L_1^* p_{n+1} = \partial_t p_n - L_2^* p_n
\end{aligned}$$

with the integralibility conditions

$$\int_0^1 p_i(t,y)dy = 0, i \geq 1 \text{ and } 1 \text{ if } i = 0.$$

# Regular Part Continued 7



- Obviously, $p_0(t, y) = p(t, y)$ is the quasi-stationary distribution of $L_1(t, x)$
- $p_{i+1}(t, y)$ is solvable because $\int_0^1 \partial_t p_i - L_2^* p_i(t, y) dy = 0$ which is the necessary and sufficient condition for Poisson equations to have a solution.
- To get the $O(\epsilon^{n+1})$ estimate, we need to solve for $p_{n+1}(t, y)$.

# Outline

# Singular Part 8

For the singular part, we have $(\partial_t - L^{\epsilon,*})(\Sigma_0^n \epsilon^i q_i(t/\epsilon, y)) = 0$. For $i = 1, 2$ and $k = 0, 1, ..., n+1$, let

$$L_i^{*,k}(0,y)f = 1/2 \cdot \partial_{yy}(\partial_{t^k}^k a_i(0,y)f) - \partial_y(\partial_{t^k}^k b_i(0,y)f).$$

Let $\tau = t/\epsilon$ and collect terms according to $\epsilon^i$, we have

$$\begin{aligned}
\partial_\tau q_0(\tau,y) &= L_1^*(0,y)q_0(\tau,y),\\
\partial_\tau q_1(\tau,y) &= L_1^*(0,y)q_1(\tau,y) + \tau L_1^{*,1}(0,y)q_0(\tau,y)\\
&\quad + L_2^*(0,y)q_0(\tau,y),\\
\partial_\tau q_2(\tau,y) &= L_1^*(0,y)q_2(\tau,y) + \tau L_1^{*,1}(0,y)q_1(\tau,y)\\
&\quad + \tau^2 L_1^{*,2}(0,y)q_0(\tau,y)\\
&\quad + L_2^*(0,y)q_1(\tau,y) + L_2^{*,1}(0,y)q_0(\tau,y),\\
&= \ldots\\
\partial_\tau q_i(\tau,y) &= L_1^*(0,y)q_i(\tau,y) + \Sigma_{j=1}^i \tau^j/j! \cdot L_1^{*,j}(0,y)q_{i-j}(\tau,y)\\
&\quad + \Sigma_{j=0}^{i-1} \tau^j/j! \cdot L_2^{*,j}(0,y)q_{i-j-1}(\tau,y),
\end{aligned}$$

# Singular Part Continued 9



- with initial conditions $q_i(0, y) = -p_i(0, y), 1 \le i \le n + 1$ and $q_0(0, y) = \delta_x(y) - p(0, y)$.
- There is a Green's function $G(x, \tau, y)$ for $\partial_\tau = L_1^*(0, y)$.
- $q_0(\tau, y) = \int_0^1 G(x, \tau, y) q_0(0, x) dx$ and
-

$$\begin{aligned} q_i(\tau, y) &= \int_0^1 G(x, \tau, y) q_i(0, x) dx \\ &+ \int_0^\tau \int_0^1 G(x, \tau - s, y) f_i(s, x) dx ds, i = 1, 2, ..., n. \end{aligned}$$

where $f_i(s, x) = \Sigma_{j=1}^i s^j / j! \cdot L_1^{*,j}(0, y) q_{i-j}(s, x) + \Sigma_{j=0}^{i-1} \tau^j / j! \cdot L_2^{*,j}(0, y) q_{i-j-1}(s, x)$.

# Exponential Decay of the Singular Part 10



- Exponential decay of $q_0(\tau, y)$.

$$
\begin{aligned}
&|q_0(\tau,y)| = |\int_0^1 G(x,\tau,y) q_0(0,x)dx| \\
&\leq sup_{x\in[0,1]}|G(x,\tau,y) - m(y)| \int_0^1 |q_0(0,x)|dx \\
&+|m(y)\int_0^1 q_0(0,x)dx| \\
&\leq K_\delta e^{-\gamma\tau}, \tau \geq \delta, \qquad \text{because}
\end{aligned}
$$

- $\int_0^1 q_0(0,x)dx = 0$ and
- $sup_{x\in[0,1]}|G(x,\tau,y) - m(y)| \leq K_\delta e^{-\gamma\tau}, \tau \geq \delta > 0.$
- For $\tau \leq \delta$, $q_0(\tau,y) \leq KG(x,\tau,y)$. Recall $q_0(0,x) = \delta_x - p(0,x)$.
- Note that $K, \delta, \gamma$ only depends on the bound of $L^\epsilon$.

# Exponential Decay of the Singular Part 11



- For general $q_i(\tau, y)$, we can also obtain the exponential bound, i.e., we have

$$|q_i(\tau, y)| \leq Ke^{-\gamma\tau}$$

for some positive constants $K, \gamma$ and $i \geq 1$.

- We have the following approximation :

$$sup_{t\leq T, y\leq 1} \quad |p^\epsilon(t,y) - (\Sigma_0^n \epsilon^i p_i(t,y) + \Sigma_0^n \epsilon^i q_{i,x}(t/\epsilon, y))| = O(\epsilon^{n+1}).$$

# Law of large numbers 12



Recall that $Z^\epsilon(t,f) = \int_0^t \left(f(X_s^\epsilon) - \int_0^1 f(y)p(s,y)dy\right) ds$. We shall show that $lim_{\epsilon\to 0} EZ^\epsilon(t,f)^2 = 0$.

$$
\begin{aligned}
Z^\epsilon\quad (t,f)^2 &= \int_0^t \int_0^t \left(f(X_r^\epsilon) - \int_0^1 f(y)p(r,x)dx\right) \\
&\left(f(X_s^\epsilon) - \int_0^1 f(x)p(s,x)dx\right) drds \\
&= \int_0^t \int_0^t \{f(X_r^\epsilon)f(X_s^\epsilon) - f(X_r^\epsilon)\int_0^1 f(x)p(s,x)dx \\
&-f(X_s^\epsilon)\int_0^1 f(x)p(r,x)dx \\
&+\int_0^1 f(x)p(r,x)dx \int_0^1 f(x)p(s,x)dx\} drds \\
&= 2\cdot \int_0^t \int_0^s \{f(X_r^\epsilon)f(X_s^\epsilon) - f(X_r^\epsilon)\int_0^1 f(x)p(s,x)dx \\
&-f(X_s^\epsilon)\int_0^1 f(x)p(r,x)dx \\
&+\int_0^1 f(x)p(r,x)dx \int_0^1 f(x)p(s,x)dx\} drds.
\end{aligned}
$$

# Law of Large numbers, Continued 13



- $Ef(X_r^\epsilon) = \int_0^1 f(x)p^\epsilon(r,x)dx = \int_0^1 f(x)P_1^\epsilon(r,x)dx + O(\epsilon^2)$ and
- $Ef(X_s^\epsilon) = \int_0^1 f(x)p^\epsilon(s,x)dx = \int_0^1 f(x)P_1^\epsilon(s,x)dx + O(\epsilon^2)$.
- For the estimate of $Ef(X_r^\epsilon)f(X_s^\epsilon)$, we have

$$\begin{aligned} Ef(X_r^\epsilon)f(X_s^\epsilon) &= E(f(X_r)E(f(X_s)|X_r)) \\ &= \int_0^1 \int_0^1 f(x)f(y)p^\epsilon(r,x)p^\epsilon(r,x;s,y)dxdy \end{aligned}$$

- $p^\epsilon(r,x;s,y)$ is the transition density of the process $X_t^\epsilon$.

# Law of Large Numbers, Continued 16



$$
\begin{aligned}
EZ^\epsilon \quad (t,f)^2 &= 2\int_0^t\int_0^s\int_0^1\int_0^1 f(x)f(y)P_1^\epsilon(r,x)(\Sigma_{i=0,1}\epsilon^i p_i(s,y) \\
&+\Sigma_{i=0,1}\epsilon^i q_{i,r,x}((s-r)/\epsilon,y)) - P_1^\epsilon(r,x)p(s,y) \\
&-P_1^\epsilon(s,y)p(r,x) + p(r,x)p(s,y))dxdydrds + O(\epsilon^2) \\
&= 2\int_0^t\int_0^s\int_0^1\int_0^1 f(x)f(y)I(x,y,r,s,\epsilon)dxdydrds + O(\epsilon^2).
\end{aligned}
$$

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# Law of Large Numbers,Continued 17



Two simple estimates : for any $\kappa > 0$,

$$\int_0^t \int_0^s e^{-(\kappa s)/\epsilon} dr ds = -(\epsilon/\kappa) t \cdot e^{-(\kappa t)/\epsilon} + \epsilon^2/\kappa^2 \cdot (1 - e^{-(\kappa t)/\epsilon})$$

$$\leq \epsilon^2/\kappa^2, \quad \text{and}$$

$$\int_0^t \int_0^s e^{-(\kappa r)/\epsilon} dr ds = \int_0^t \int_0^s e^{-\kappa(s-r)/\epsilon} dr ds$$

$$= \int_0^t \epsilon/\kappa \cdot (1 - e^{-(\kappa s)/\epsilon}) ds \leq (\epsilon/\kappa) t.$$

# Law of Large Numbers,Continued 18



We now simplify the term $I(x, y, r, s, \epsilon)$ as follows.

$$I = P_1^\epsilon(r,x)(\Sigma_{i=0}^1 \epsilon^i p_i(s,y) + \Sigma_{i=0}^1 \epsilon^i q_{i,r,x}((s-r)/\epsilon, y))$$
$$-P_1^\epsilon(r,x)p(s,y) - (\Sigma_{i=0}^1 \epsilon^i p_i(s,y) + \Sigma_{i=0}^1 \epsilon^i q_i(s/\epsilon, y))p(r,x)$$
$$+p(r,x)p(s,y) + O(\epsilon^2)$$

$$= (\Sigma_{i=0,1} \epsilon^i p_i(r,x) + \Sigma_{i=0,1} \epsilon^i q_i(r/\epsilon, x))$$
$$\cdot(\epsilon p_1(s,y) + \Sigma_{i=0,1} \epsilon^i q_{i,r,x}((s-r)/\epsilon, y))$$
$$-(\epsilon p_1(s,y) + \Sigma_{i=0,1} \epsilon^i q_i(s/\epsilon, y))p(r,x) + O(\epsilon^2)$$

$$= (\epsilon p_1(r,x) + \Sigma_{i=0,1} \epsilon^i q_i(r/\epsilon, x))$$
$$\cdot(\epsilon p_1(s,y) + \Sigma_{i=0,1} \epsilon^i q_{i,r,x}((s-r)/\epsilon, y))$$
$$+p(r,x)(\Sigma_{i=0}^1 \epsilon^i q_{i,r,x}((s-r)/\epsilon, y) - \Sigma_{i=0}^1 \epsilon^i q_i(s/\epsilon, y)) + O(\epsilon^2)$$

$$= q_0(r/\epsilon, x) q_{0,r,x}((s-r)/\epsilon, y) + p(r,x) q_{0,r,x}((s-r)/\epsilon, y)$$
$$+O(\epsilon)(e^{-\gamma r/\epsilon} + e^{-\gamma(s-r)/\epsilon}) + O(\epsilon^2)$$

# Law of Large Numbers,Continued 19

Hence,

$$\int_0^t \int_0^s I(x,y,r,s,\epsilon)drds$$

$$= \int_0^t \int_0^s p(r,x)q_{0,r,x}((s-r)/\epsilon, y) + O(e^{-\gamma s/\epsilon}) \\ +O(\epsilon)O(e^{-\gamma r/\epsilon}) + O(\epsilon^2)drds$$

$$= O(\epsilon^2) + \int_0^t \int_0^s p(r,x)q_{0,r,f_r^\epsilon}((s-r)/\epsilon, y)drds.$$

Finally,

$$EZ^\epsilon(t,f)^2 \\ = 2\int_0^t \int_0^s \int_0^1 \int_0^1 f(x)f(y)p(r,x)q_{0,r,x}((s-r)/\epsilon, y)drdsdxdy \\ +O(\epsilon^2).$$

# Law of Large Numbers,Continued 20



- $$EZ^\epsilon(t,f)^2$$
  $$= 2\epsilon \int_0^1 \int_0^1 \int_0^t \int_0^\infty f(x)f(y)p(r,x)q_{0,r,x}(u,y)dudrdxdy + O(\epsilon^2).$$
- Obviously, $lim_{\epsilon\to 0} EZ^\epsilon(t,f)^2 = 0$, thus the law of large of numbers holds.

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# Outline

- A stochastic process $X_t$ is $\phi$-mixing if for any $t, s$, $A \in \mathcal{F}_0^t$ and $B \in \mathcal{F}_{t+s}^\infty$, we have

$$|P(B|A) - P(B)| \le \phi(s).$$

- The process $X_t^\epsilon$ satisfies the mixing condition with mixing rate $K \cdot e^{-(\kappa/\epsilon)s}$, i.e., for any $\eta \in \mathcal{F}_{s+t}^\infty, |\eta| \le 1$,

$$|E(\eta|\mathcal{F}_0^t) - E\eta| \le K \cdot e^{-(\kappa/\epsilon)s}.$$

- Thus for any random variable $\xi \in \mathcal{F}_0^t$ and $|\xi| \le 1$, we have

$$|E(\xi\eta) - E\xi \cdot E\eta| \le K \cdot e^{(-\kappa/\epsilon)s}.$$

# Outline

# Tightness 22



- Let $\bar{n}^\epsilon(t, f) = 1/\sqrt{\epsilon}\int_0^t \Big( f(X_s^\epsilon) - \int_0^1 f(y)p^\epsilon(s, y)dy \Big) ds$
- since

$$\begin{aligned} n^\epsilon(t, f) = \;& \bar{n}^\epsilon(t, f) \\ & +1/\sqrt{\epsilon}\int_0^t \Big( Ef(X_t^\epsilon) - \int_0^1 f(y)p(s, y)dy \Big) ds \\ & = \bar{n}^\epsilon(t, f) + 1/\sqrt{\epsilon} EZ^\epsilon(t, f) \\ & = \bar{n}^\epsilon(t, f) + O(\sqrt{\epsilon}), \end{aligned}$$

the tightness of $n^\epsilon(\cdot, f)$ will follow from that of $\bar{n}^\epsilon(\cdot, f)$.

- We establish the tightness of $\bar{n}^\epsilon(\cdot, f)$ by showing that $E(\bar{n}^\epsilon(t+s, f) - \bar{n}^\epsilon(t, f))^4 \leq Ks^2$ for any $s, t > 0$.

# Tightness 23



Let $h(r) = f(X_r^\epsilon) - Ef(X_r^\epsilon)$. Since $\bar{n}^\epsilon(t+s,f) - \bar{n}^\epsilon(t,f) = 1/\sqrt{\epsilon}\int_t^{t+s} h(r)dr$, we have

$$
\begin{aligned}
&E(\bar{n}^\epsilon(t+s,f) - \bar{n}^\epsilon(t,f))^4 \\
&= 1/\epsilon^2 \int_t^{t+s}\int_t^{t+s}\int_t^{t+s}\int_t^{t+s} Eh(r_1)h(r_2)h(r_3)h(r_4)dr_1..dr_4 \\
&= 24/\epsilon^2 \int_D Eh(r_1)h(r_2)h(r_3)h(r_4)dr_1..dr_4
\end{aligned}
$$

where $D = \{(r_1,..,r_4), t \le r_1 \le r_2 \le r_3 \le r_4 \le t+s\}$.

# Tightness 24



Note that

$$\begin{aligned}
&|Eh(r_1)h(r_2)h(r_3)h(r_4)| \\
&\leq |Eh(r_1)h(r_2)h(r_3)h(r_4) - Eh(r_1)h(r_2)\cdot Eh(r_3)h(r_4)| \\
&+|Eh((r_1)h(r_2))|\cdot|Eh(r_3)h(r_4)| \\
&\leq |Eh(r_1)h(r_2)h(r_3)h(r_4) - Eh(r_1)h(r_2)\cdot Eh(r_3)h(r_4)| \\
&+Ke^{-\kappa(r_2-r_1)/\epsilon}\cdot Ke^{-\kappa(r_4-r_3)/\epsilon}.
\end{aligned}$$

The last inequality follows from the mixing condition and $Eh(r)=0$. Similarly,

$$|Eh(r_1)h(r_2)h(r_3)h(r_4)-Eh(r_1)h(r_2)\cdot Eh(r_3)h(r_4)| \leq Ke^{-\kappa(r_3-r_2)/\epsilon}.$$

# Tightness 25



Hence,

$$\begin{aligned}
&|Eh(r_1)h(r_2)h(r_3)h(r_4) - Eh(r_1)h(r_2)\cdot Eh(r_3)h(r_4)| \\
&= (|Eh(r_1)h(r_2)h(r_3)h(r_4) - Eh(r_1)h(r_2)\cdot Eh(r_3)h(r_4)|^{1/2})^2 \\
&\leq Ke^{-\kappa(r_3-r_2)/(2\epsilon)}\cdot(|Eh(r_1)h(r_2)h(r_3)h(r_4)|^{1/2} \\
&+|Eh(r_1)h(r_2)\cdot Eh(r_3)h(r_4)|^{1/2})
\end{aligned}$$

The last inequality follows from $\sqrt{a-b}\leq\sqrt{a}+\sqrt{b}$. Also,

$$\begin{aligned}
&|Eh(r_1)h(r_2)h(r_3)h(r_4)| \\
&= |Eh(r_1)h(r_2)h(r_3)E(h(r_4)|\mathcal{F}_0^{r_3}) - Eh(r_4)| \\
&\leq M^3Ke^{-\kappa(r_4-r_3)/\epsilon}.
\end{aligned}$$

Here, $M = sup_x|f(x)|$.

# Tightness 26



And

$$\begin{aligned}
&|Eh(r_1)h(r_2) \cdot Eh(r_3)h(r_4)| \\
&= |Eh(r_1)h(r_2) - Eh(r_1)Eh(r_2)||Eh(r_3)h(r_4) - Eh(r_3)Eh(r_4)| \\
&\leq K^2 e^{-\kappa(r_2-r_1)/\epsilon} e^{-\kappa(r_4-r_3)/\epsilon}
\end{aligned}$$

Thus

$$\begin{aligned}
&|Eh(r_1)h(r_2)h(r_3)h(r_4) - Eh(r_1)h(r_2) \cdot Eh(r_3)h(r_4)| \\
&\leq Ke^{-\kappa(r_3-r_2)/2\epsilon} \\
&\cdot(e^{-\kappa(r_2-r_1)/2\epsilon} e^{-\kappa(r_4-r_3)/2\epsilon} + e^{-\kappa(r_4-r_3)/2\epsilon}) \\
&= K(e^{-\kappa(r_4-r_1)/2\epsilon} + e^{-\kappa(r_4-r_2)/2\epsilon}).
\end{aligned}$$

# Tightness 27



Two simple estimates,

$$\int_D e^{-\kappa(r_2-r_1)/\epsilon} e^{-\kappa(r_4-r_3)/\epsilon} dr_1 dr_2 dr_3 dr_4$$
$$\leq \int_t^{t+s} s\epsilon^2/\kappa^2(1 - e^{-\kappa(r_4-t)/\epsilon})dr_4 \leq s^2\epsilon^2/\kappa^2$$

and

$$\int_D e^{-\kappa(r_4-r_1)/2\epsilon} dr_1 dr_2 dr_3 dr_4 \leq \int_D e^{-\kappa(r_4-r_2)/2\epsilon} dr_1 dr_2 dr_3 dr_4$$
$$\leq 4s^2\epsilon^2/\kappa^2.$$

We therefore conclude that $E(\bar{n}^\epsilon(t+s, f) - \bar{n}^\epsilon(t, f))^4 \leq Ks^2$ for every $\epsilon$ and $\bar{n}^\epsilon(\cdot, f)$ is tight in $C[0, T]$.

# Outline

Consider $n(t_4,f)-n(t_3,f)$ and $n(t_2,f)-n(t_1,f)$ where $t_4>t_3>t_2>t_1\geq 0$. By the mixing property, we have

$$\begin{aligned}&\Big|Ee^{iu_1(n^\epsilon(t_4,f)-n^\epsilon(t_3,f))+iu_2(n^\epsilon(t_2,f))-n^\epsilon(t_1,f))}\\&-Ee^{iu_1(n^\epsilon(t_4,f)-n^\epsilon(t_3,f))}Ee^{iu_2(n^\epsilon(t_2,f)-n^\epsilon(t_1,f))}\Big|\\&\leq Ke^{-\kappa(t_3-t_2)/\epsilon}\longrightarrow 0 \quad\text{as}\quad \epsilon\to 0\end{aligned}$$

for any $u_1,u_2\in R$.

But

$$
\begin{aligned}
&\Big| E e^{iu_1(n^\epsilon(t_4,f)-n^\epsilon(t_3,f))+iu_2(n^\epsilon(t_2,f)-n^\epsilon(t_1,f))} \\
&- E e^{iu_1(n^\epsilon(t_4,f)-n^\epsilon(t_3,f))} E e^{iu_2(n^\epsilon(t_2,f)-n^\epsilon(t_1,f))} \Big| \\
&\longrightarrow \Big| E e^{iu_1(n(t_4,f)-(t_3,f))+iu_2(n(t_2,f)-n(t_1,f))} \\
&- E e^{iu_1(n(t_4,f)-n(t_3,f))} E e^{iu_2(n(t_2,f)-n(t_1,f))} \Big| \quad \text{as} \quad \epsilon \longrightarrow 0,
\end{aligned}
$$

we thus have

$$
\begin{aligned}
&E e^{iu_1(n(t_4,f)-n(t_3,f))+iu_2(n(t_2,f))-n(t_1,f))} \\
&= E e^{iu_1(n(t_4,f)-n(t_3,f))} E e^{iu_2(n(t_2,f)-n(t_1,f))}.
\end{aligned}
$$

This shows the independence of $n(t_4,f) - n(t_3,f)$ and $n(t_2,f) - n(t_1,f)$. Let $t_3 \longrightarrow t_2$ and then complete the proof.

# Covariance 30



Continuous sample paths and independent increments imply that $n(t,f)$ is a Gaussian process (with mean 0). We now compute its covariance. Since

$$EZ^\epsilon(t,f)^2$$
$$= 2\epsilon \int_0^1 \int_0^1 \int_0^t \int_0^\infty f(x)f(y)p(r,x)q_{0,r,x}(u,y)dudrdxdy$$
$$+O(\epsilon^2),$$

we obviously have that for two non-negative functions $f$ and $g$,

$$EZ^\epsilon(t,f)\cdot Z^\epsilon(t,g)$$
$$= \epsilon \int_0^1 \int_0^1 \int_0^t \int_0^\infty f(x)g(y)p(r,x)q_{0,r,x}(u,y)dudrdxdy$$
$$+\epsilon \int_0^1 \int_0^1 \int_0^t \int_0^\infty f(y)g(x)p(r,x)q_{0,r,x}(u,y)dudrdxdy$$
$$+O(\epsilon^2).$$

It follows that

# Covariance 31



$$lim_{\epsilon\to 0} En^\epsilon(t,f)^2$$
$$= 2\int_0^1\int_0^1\int_0^t\int_0^\infty f(x)f(y)p(r,x)q_{0,r,x}(u,y)dudrdxdy$$
$$= En(t,f)^2.$$

The covariance fuction of $n(t,f)$ is thus $En(t,f)n(s,f) = En(min\{t,s\},f)^2$.

# References 32 I



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