The Support Properties for Λ-Fleming-Viot Processes

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Outline of the Talk

Introduction to Fleming-Viot process

- The Fleming-Viot process is a probability-measure-valued Markov processes for mathematical population genetics.
- It describes the evolution of relative frequencies for different types of alleles in a large population.
- In this talk we consider the Fleming-Viot processes undergoing resampling and Brownian mutation.
- The classical Fleming-Viot process is dual to the Kingman's coalescent. The Λ-Fleming-Viot process is dual to the more general Λ-coalescent.

The state space of coalescents

- $[n] = \{1, \dots, n\}$
- $[\infty] = \{1, 2, \dots\}$
- A partition $\pi = \{\pi_i, i = 1, 2, \dots\}$ of $D \subset [\infty]$ is a collection of disjoint blocks such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$.
- $\mathcal{P}_n$: the set of partitions of $[n]$.
- $\mathcal{P}_\infty$: the set of partitions of $[\infty]$.

Kingman's coalescent

- Given that there are n blocks, each 2-tuple of blocks merges independently to form a single block at rate 1.
- The transition rate from n blocks to $n-1$ blocks is $n(n-1)/2$.
- Note that only binary mergers are allowed. With the transition rate defined above, the $\mathcal{P}_n$-valued Markov process $\{\Pi_n(t) : t \geq 0\}$ is called an n-coalescent.
- Kingman (1982) shows that there exists a $\mathcal{P}_\infty$-valued Markov process $\{\Pi(t) : t \geq 0\}$ which is called Kingman's coalescent, and whose restriction to the first n positive integers is an n-coalescent.

Λ-coalescent

- Pitman (1999) and Sagitov (1999) introduces the Λ-coalescent which allows multiple collisions.
- It is a $\mathcal{P}_\infty$-valued Markov process such that given there are n blocks in the partition, each k-tuple of blocks ($2 \le k \le n$) independently merges to form a single block at rate

$$\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda(dx)$$

and Λ is a finite measure on $[0,1]$.

- Consistent condition: for all $2 \le k \le n$:

$$\lambda_{n,k} = \lambda_{n+1,k} + \lambda_{n+1,k+1}.$$

Coming down from infinity

Let $\#\Pi_\infty(t)$ be the number of blocks in the partition $\Pi_\infty(t)$.

- The coalescent process comes down from infinity if

$$P\left(\#\Pi_\infty(t) < \infty\right) = 1$$

  for all $t > 0$.

- The coalescent process stays infinite if

$$P\left(\#\Pi_\infty(t) = \infty\right) = 1$$

  for all $t > 0$.

Coming down from infinity

- Schweinsberg (2000) Suppose that $\Lambda(\{1\}) = 0$. The Λ-coalescent comes down from infinity if and only if
  - either $\Lambda(\{0\}) > 0$;
  - or $\Lambda(\{0\}) = 0$ but

$$\sum_{n=2}^{\infty}\left(\sum_{k=2}^{n}(k-1)\binom{n}{k}\int_{(0,1]}x^{k-2}(1-x)^{n-k}\Lambda(dx)\right)^{-1} < \infty.$$

Examples

- If $\Lambda = \delta_0$, the corresponding coalescent degenerates to Kingman's coalescent and comes down from infinity.
- If $\Lambda = \delta_1$, the corresponding coalescent neither comes down from infinity nor stays infinite.
- If Λ is the uniform distribution on $[0, 1]$, the corresponding coalescent is called the U-coalescent, which does not come down from infinity.

Examples

- If
$$\Lambda(dx) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} (1-x)^{\beta-1} dx$$
for some $\beta \in (0, 2]$, the corresponding coalescent is called the Beta$(2-\beta, \beta)$-coalescent.
    - When $\beta \in (0, 1)$, it does not come down from infinity;
    - When $\beta = 1$, it is the U-coalescent;
    - When $\beta \in (1, 2]$, it comes down from infinity.
- If $\Lambda(dx) \geq x^{-\alpha} dx,\ 0 < x < \epsilon$ for some $\alpha \in (0, 1)$ and $\epsilon \leq 1$, the corresponding coalescent is called the $\Lambda(\epsilon, \alpha)$-coalescent and it comes down from infinity.

Key features of the lookdown construction

- The lookdown particle construction of Donnelly and Kurtz is a powerful tool to study the FV processes.
- In this particle system $(X_1(t), X_2(t), X_3(t), \cdots)$, particle $X_i(t)$ is attached a "level" i.
- Looking forwards in time, the empirical measure of the countably infinite particles in the lookdown model converges to the FV processes;
- Looking backwards in time, we can recover the coalescent processes from the lookdown model that describes the genealogical structure of the particles.
- The evolution of a particle at level n only depends on the evolution of the particles with lower levels. This property allows us to construct approximating particle systems, and their limit in the same probability space.

- For any finite measure Λ on $[0, 1]$, we have $\Lambda = a\delta_0 + \Lambda_0$, where $a\delta_0$ is the restriction of Λ to $\{0\}$ and $\Lambda_0 = \Lambda \mathbf{1}_{(0,1]}$.
- The particle system undergoes reproductions (resampling). The particles move independently between the reproduction events.
- There are two kinds of reproduction events
  - single birth event associated to $a\delta_0$;
  - multiple birth event associated to Λ_0.

Lookdown construction with single birth

- Let $\{N_{ij}(t) : 1 \leq i < j < \infty\}$ be independent Poisson processes with common rate a.
- At a jump time t of N_{ij}, i.e. $\Delta N_{ij}(t) = 1$:
  - the particle at level j looks down at level i and assumes the spatial location of particle at level i, corresponding to a single birth event;
  - all the other particles with levels above j are shifted upwards.
  - We have

$$X_k(t) = \begin{cases} X_k(t-), & \text{if } k < j, \\ X_i(t-), & \text{if } k = j, \\ X_{k-1}(t-), & \text{if } k > j. \end{cases}$$

Lookdown construction with multiple birth

- Let **N** be a Poisson point process on $\mathbb{R}_+ \times (0,1]$ with intensity measure $dt \otimes x^{-2}\Lambda_0(dx)$.
- Let $\{U_{ij},\ i,j \in [\infty]\}$ be i.i.d. uniform $[0,1]$ distribution.
- Jumps (t_i, x_i) of **N** correspond to multiple birth events.
- Intuitively, at a jump time t_i, all the particles at levels j with $U_{ij} \leq x_i$ participate in the birth event.
- The particles involved in the multiple birth event assume the spatial location of the particle with the lowest level involved.
- The particles at other levels, keeping their original order, are shifted upwards accordingly.

Lookdown construction for Λ-FV with Brownian mutation

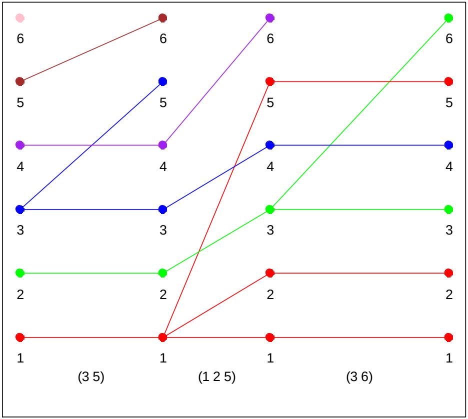
- At time $t = t_i$, if j is the lowest level involved, then

$$X_k(t) = \begin{cases} X_k(t-), & \text{for } k \le j, \\ X_j(t-), & \text{for } k > j \text{ with } U_{ik} \le x_i, \\ X_{k-J_t^k}(t-), & \text{otherwise,} \end{cases}$$

  where $J_{t_i}^k := \#\{m < k, U_{im} \le x_i\} - 1$.
- Between jump times, particles at different levels move according to independent Brownian mutations.

Lookdown Construction

There are three lookdown events involved.



Λ-coalescent in the lookdown construction

- Suppose that $(X_i(0))$ is exchangeable.
- Then for each $t > 0$, $(X_i(t))$ is exchangeable, so that

$$X(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}$$

exists almost surely by the de Finetti's theorem, and is the Λ-Fleming-Viot process with Brownian mutation (cf. Donnelly and Kurtz (1999)).

Λ-coalescent in the lookdown construction

- The lookdown construction induces a genealogical structure to the particles.
- For $s \leq t$, let $L_n^t(s)$ be the ancestor level at time s of the particle with level n at time t.
- For fixed $T > 0$, denote $\{\Pi(t) : 0 \leq t \leq T\}$ as a $\mathcal{P}_\infty$-valued process such that i and j are in the same block of $\Pi(t)$ if and only if $L_i^T(T-t) = L_j^T(T-t)$.
- The process $\{\Pi(t) : 0 \leq t \leq T\}$ is a Λ-coalescent process (cf. Donnelly and Kurtz (1999)).

Theorem (Dawson and Hochberg (1982))

Let $X(t)$ be the classical Fleming-Viot process in $\mathbb{R}^d$ such that $X(0)$ has a compact support. Then for any fixed $t > 0$, $X(t)$ has compact support and its support has Hausdorff dimension bounded from above by two.

- Remark: $X(0)$ does not need to have a compact support.

Previous results

Theorem (Blath 2009)

Let $\{\Pi(t) : t \geq 0\}$ *be the* Λ*-coalescent and* $\{X(t) : t \geq 0\}$ *be the corresponding* Λ*-FV process with Brownian mutation. If* $\{\Pi(t) : t \geq 0\}$ *does not come down from infinity, then for each* $t > 0$,

$$Supp\,(X(t)) = \mathbb{R}^d.$$

Main result (simple version)

Theorem (Liu and Z.)

If there exist constants $C > 0$ and $\alpha > 0$ such that the total coalescent rate

$$\lambda_b = \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k}$$

of the associated Λ-coalescent satisfies

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \frac{C}{m^\alpha}$$

for m big enough. Then for any $t > 0$, the Λ-Fleming-Viot process has a compact support at t. In addition

$$dim(supp(X_t)) \leq 2/\alpha.$$

Lower bound of Hausdorff dimension

Lemma

If there is a measure μ *supported by* $A \subset \mathbb{R}$ *such that*

$$\int\int \frac{1}{|x-y|^a}\mu(dx)\mu(dy) < \infty,$$

then $dim(A) \geq a$.

Proposition

$$dim(supp(X_t)) \geq 2.$$

Corollary

If $\Lambda(\{0\}) > 0)$, *then*

$$dim(supp(X_t)) = 2.$$

Beta$(2-\beta,\beta)$-Fleming-Viot process

Corollary

For any $t>0$ the Beta$(2-\beta,\beta)$-Fleming-Viot process X has a compact support at time t if and only if $\beta\in(1,2)$. Further,

$$dim(supp(X_t)) \le 2/\left(\beta-1\right)$$

for $\beta\in(1,2)$.

Corollary

For any $t>0$ the $\Lambda(\epsilon,\alpha)$-Fleming-Viot process X has a compact support at time t with

$$dim(supp(X_t)) \le 2/\alpha.$$

Intuition

Intuitively, if the dual Λ-coalescent comes down from infinity, the particles alive at time t belong to finitely many families at time $t-\epsilon$ for any $\epsilon > 0$. The particles from the same family are highly corelated and should stay close to each other to form a cluster. We can find a small ball to cover each cluster.

Outline of the proof

- Choose a geometric sequence $a_n \uparrow \infty$ and estimate T_n, the time for the Λ-coalescent recovered from the lookdown construction to come down to below a_n.
- Group the individuals alive at time T_{n+1} by their ancestors at time T_n. Estimate the maximal dislocation of the individuals alive at time T_{n+1} from their respective ancestors at time T_n.
- Show that the maximal dislocations are summable.
- Such a procedure induces a random cover on the support, which gives an upper bound on the Hausdorff dimension.

Further work

- Exact Hausdorff dimension of the support.
  The upper bound on the Hausdorff dimension appears to be sharp.

- Modulus of continuity of the support.
  Find function f so that

$$supp(X_{t+\epsilon}) \subset B_{f(\epsilon)}(supp(X_t)),$$

where $B_{f(\epsilon)}(supp(X_t))$ is the $f(\epsilon)$-neighborhood of $supp(X_t)$.

Thanks for your attention.
Questions?

The (Λ, A)-FV process

Definition

The (Λ, A)-FV process is a probability-measure-valued Markov process $(X(t))_{t\geq 0}$ with generator,

$$\mathcal{L}F(\mu) \equiv \sum_{i=1}^{n} \langle \mu, Af_i \rangle \prod_{j \neq i} \langle \mu, f_j \rangle$$

$$+ \sum_{J \subset \{1, \cdots, n\}, |J| \geq 2} \lambda_{n,|J|} \left[\left\langle \mu, \prod_{j \in J} f_j \right\rangle - \prod_{j \in J} \langle \mu, f_j \rangle \right] \prod_{k \notin J} \langle \mu, f_k \rangle ,$$

where A is the generator of a Feller process and

$$F(\mu) = \prod_{i=1}^{n} \langle \mu, f_i \rangle \quad \text{with } f_i \in \mathcal{D}(A) \text{ for } i = 1, 2, \cdots, n.$$

Hausdorff dimension

- (U_i) is a δ-cover of A if
  - $$A \subset \cup_{i=1}^{\infty} U_i,$$
  - $$\sup_i |U_i| < \delta,$$

  where $|U_i|$ denotes the diameter of U_i.
- let $H^s_\delta(A) = \inf \sum_{i=1}^{\infty} |U_i|^s$ and $H^s(A) = \lim_{\delta\to 0} H^s_\delta$.
- The Hausdorff dimension of A is the unique value, denoted by $dim(A)$, such that

$$dim(A) = \inf\{s : H^s(A) = 0\} = \sup\{s : H^s(A) = \infty\}.$$