Exact Stationary Tail Asymptotics for Two-Dimensional Reflecting Brownian Motion — A Kernel Method

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(Based on joint work with Hongshuai Dai)

Outline

SRBM model

- Two-dimensional reflecting BM with state space $\mathbb{R}_+^2$:

$$Z(t) = X(t) + RY(t)$$

where

- X is a Brownian motion with drift vector μ and covariance matrix Σ
- Y is continuous and non-decreasing with $Y(0) = 0$, and Y_j only increase at times t for which $Z_j(t) = 0$, $j = 1, 2$
- R is a 2×2 matrix specifying the direction of the increase for Y

Existence of SRBM and its stationary distribution

- R is an $\mathbb{S}$-matrix, if there exists a 2-vector $\omega \geq 0$ such that $R\omega \geq 0$. R is completely-$\mathbb{S}$, if each of its principal sub-matrices is an $\mathbb{S}$-matrix.
- For a given data set (Σ, μ, R) with Σ being positive definite, there exists an SRBM for each initial distribution of $Z(0)$ if and only if R is completely-$\mathbb{S}$. Furthermore, when R is completely-$\mathbb{S}$, the SRBM is unique in distribution for each given initial distribution.
- A necessary and sufficient condition for $Z(t)$ to have a stationary distribution is that R is a $\mathbb{P}$-matrix (all of its principal minors are positive) and R is non-singular with $R^{-1}\mu < 0$.

Boundary measures

- We assume that the SRBM has a unique stationary distribution π. Also assume that $Z(0)$ follows the stationary distribution.
- From the literature result, we get that each component of $\mathbb{E}_\pi(Y(1))$ is finite. Therefore, we can define

$$V_i(A) = \mathbb{E}_\pi\left[\int_0^1 1_{\{Z(u)\in A\}} dY_i(u)\right], i = 1, 2,$$

  where $A \subset \mathbb{R}^2_+$ is a Borel set. V_i is a finite measure on $\mathbb{R}^2_+$ with the support on the boundary $\{x \in \mathbb{R}^2_+ : x_i = 0\}$.
- Our focus in on the tail behavior of the boundary measures $V_i, i = 1, 2$.

Types of tail asymptotics

- (Rough asymptotics) Let $g(x)$ be a positive valued function of $x \in [0, \infty)$. If
$$\alpha = \lim_{x\to\infty} -\frac{1}{x} \log g(x)$$
exists, $g(x)$ is said to have a rough decay rate α.
- On the other hand, if there exists a function h such that
$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1,$$
then $g(x)$ is said to have exact asymptotics $h(x)$.

Originated from combinatorics

- The original idea was first introduced by Knuth [5] and later developed as the kernel method by Banderier *et al.* [1].
- The original method deals with the fundamental form:

$$K(x,y)F(x,y) = A(x,y)G(x) + B(x,y),$$

  where $F(x,y)$ and $G(x)$ are unknown functions.
- The key idea in the kernel method is to find a branch $y = y_0(x)$, such that, at $(x, y_0(x))$, the kernel function is zero, or $K(x, y_0(x)) = 0$.
- When analytically substituting this branch into the RHS of the FM, we have $G(x) = -B(x, y_0(x))/A(x, y_0(x))$, and hence,

$$F(x,y) = \frac{-A(x,y)B(x,y_0(x))/A(x,y_0(x)) + B(x,y)}{K(x,y)}.$$

FM for SRBM model

- Let $Z = (Z_1, Z_2)$ be a random vector following the stationary distribution.
- For $\hat{x} = (x, y) \in \mathbb{R}^2$, define three MG functions:

$$\phi(x,y) = \mathbb{E}\big(e^{<\hat{x},Z>}\big),$$

$$\phi_1(y) = \int_{\mathbb{R}^+} e^{yw_2} V_1(dw) = \mathbb{E}_\pi \int_0^1 e^{yZ_2(u)} dY_1(u),$$

$$\phi_2(x) = \int_{\mathbb{R}^+} e^{xw_1} V_2(dw) = \mathbb{E}_\pi \int_0^1 e^{\theta_1 Z_1(u)} dY_2(u).$$

- Let $R = \big(r_{ij}\big)_{2\times 2}$ and $\Sigma = \big(\Sigma_{ij}\big)_{2\times 2}$. It follows from Dai and Miyazawa [4] that the FM for SRBM is given by

$$\gamma(x,y)\phi(x,y) = \gamma_1(x,y)\phi_1(y) + \gamma_2(x,y)\phi_2(x),$$

FM for SRBM model

where

$$\gamma_1(x,y) = r_{11}x + r_{21}y,$$
$$\gamma_2(x,y) = r_{12}x + r_{22}y,$$
$$\gamma(x,y) = - < \hat{x}, \mu > - \frac{1}{2} < \hat{x}, \Sigma\hat{x} > .$$

- However, when applying the key idea of the original kernel method to the FM of the SRBM, we only have a relationship between the two unknown functions $\phi_1(y)$ and $\phi_2(x)$.
- Therefore, a good understanding on the interlace of these two functions is crucial.

Extension of the kernel method

- There is no need to express the unknown MG functions first for the purpose of characterizing tail asymptotics.
- We do need the information about the location of the dominant singularity of the unknown functions and the detailed asymptotic property of the function at the dominant singularity.
- A Tauberian-like theorem for the MG function will be developed for linking the asymptotic property of the unknown function at the dominant singularity to the tail asymptotic property of the boundary measure.

Branch points

Consider the kernel equation:

$$\gamma(x,y) = \frac{1}{2}\Sigma_{22}y^2 + (\mu_2 + \Sigma_{12}x)y + \frac{1}{2}\Sigma_{11}x^2 + x\mu_1$$
$$= ay^2 + b(x)y + c(x) = 0.$$

Let $D_1(x) = b^2(x) - 4ac(x)$. Then, for each fixed x, the two solutions for y are given by

$$Y_\pm(x) = \frac{-b(x) \pm \sqrt{b^2(x) - 4ac(x)}}{2a},$$

unless $D_1(x) = 0$, for which the two solutions coincide and x is called a branch point of Y.
Symmetrically, we can consider the two solutions for x for each fixed y.

Branch points

We have the following properties on the branch points.

Lemma

$D_1(x)$ has two zeros satisfying $x_1 \leq 0 < x_2$. Furthermore, $D_1(x) > 0$ in (x_1, x_2).

Similarly, $D_2(y)$ has two zeros satisfying $y_1 \leq 0 < y_2$. Moreover, $D_2(y) > 0$ in (y_1, y_2).

Branches — analytic continuation

- Y_- and Y_+ are analytic in a region in $\mathbb{C}_x$, respectively.
- For continuation of them, consider the cut plan

$$\widetilde{\mathbb{C}}_x = \mathbb{C}_x - (-\infty, x_1] \cup [x_2, \infty).$$

- However, $Y_-(x)$ or Y_+ cannot be analytic (or meromorphic) in the cut plan.
- Define

$$Y_0(x) = \begin{cases} Y_-(x), & \text{if } |Y_-(x)| \le |Y_+(x)|, \\ Y_+(x), & \text{if } |Y_-(x)| > |Y_+(x)|; \end{cases}$$

Branches — analytic continuation

Lemma

$Y_0(x)$ is analytic on $\widetilde{\mathbb{C}}_x$.

Based on the property of $Y_0(x)$, we have

Lemma

The function $\gamma_2(x, Y_0(x))$ is analytic on $\widetilde{\mathbb{C}}_x$. Similarly, the function $\gamma_1(X_0(y), y)$ is analytic on $\widetilde{\mathbb{C}}_y$.

Asymptotic analysis of $\phi_1(y)$ and $\phi_2(x)$

Asymptotic properties for $\phi_1(y)$ and $\phi_2(x)$ are the key for characterizing exact tail asymptotics for the two boundary distributions $V_1(y)$ and $V_2(x)$. For this purpose, we do the following:

Lemma

Let $g(\lambda) = \int_0^\infty e^{\lambda x} dF(x)$ be the moment generating function of a probability distribution F on $\mathbb{R}_+$ with real variable λ. Define the convergence parameter of g as

$$C_p(g) = \sup\{\lambda \geq 0 : g(\lambda) < \infty\}.$$

Then the complex variable function $g(z)$ is analytic on $\{z \in \mathbb{C}_z; Re(z) < C_p(g)\}$. It implies that $C_p(g)$ is a singular point for the complex $g(z)$.

From above Lemma, we get that

Lemma

$\phi_1(z)$ is analytic on $\{z : Re(z) < \tau_2\}$, and $\phi_2(z)$ is analytic on $\{z : Re(z) < \tau_1\}$, where $C_p(\phi_1) = \tau_2$, and $C_p(\phi_2) = \tau_1$.

The following lemma characterizes τ_1 and τ_2.

Lemma

$\tau_2 > 0$ and $\tau_1 > 0$.

A singular point x_{dom} satisfying $Re(x_{dom}) = \tau_1$ is called a dominant singularity for $\phi_2(z)$. There are three candidates for a dominant singularity according to the following theorem.

Theorem

If $Re(x_{dom}) \neq x_2$, then either $x_{dom} = x^*$ is a zero of $\gamma_2(x, Y_0(x))$ or $x_{dom} = \widetilde{x}$ satisfying $Y_0(\widetilde{x})$ a zero of $\gamma_1(X_0(y), y)$. In each case, x_{dom} is real.

Theorem

In any case, there is only one dominant singularity.

Corollary

Except at τ_1, $\phi_2(x)$ can be analytically continued to $Re(z) = \tau_1 + \varepsilon$; or for some $0 < \delta < \pi/2$, $\phi_2(x)$ is analytic in the dented region:

$$G(\delta, \varepsilon) = \{z \in \mathbb{C}_z : Re(z) < 1 + \varepsilon, z \neq \tau_1, |arg(z - \tau)| > \delta\}.$$

Tauberian-like theorem

Let V be a r.v. having distribution $V(t)$ and $v(s) = \mathbb{E}(e^{sV})$. The following Tauberian-like theorem, which is an extension of Gustanv [4], links the asymptotic property of $v(s)$ at its its dominant singularity to the tail property of $V(t)$.

Theorem

Assume that $v(s)$ has only one dominant singularity α_0 and has the following asymptotic property at α_0 for $\lambda_0 \notin \{0, 1, 2, \ldots\}$,

$$\lim_{s \to \alpha_0} (s - \alpha_0)^{-\lambda_0} v(s) = c_0 \neq 0,$$

where the limit is taken in a dented neighborhood of α_0. Then, $V(t)$ has the following tail asymptotic property, for large t:

$$V(t) \sim e^{-\alpha_0 t} \frac{c_0}{\Gamma(-\lambda_0)} t^{-\lambda_0 - 1}.$$

Asymptotic property at x_{dom}

Theorem

For the function $\phi_2(x)$, a total of four types of asymptotics exists:

Case 1. *If* $\tau_1 = x^* < \min\{\tilde{x}, x_2\}$, *or* $\tau_1 = \tilde{x} < \min\{x^*, x_2\}$, *or* $\tau_1 = \tilde{x} = x_2$, *then*

$$\lim_{x \to \tau_1} (\tau_1 - x)\phi_2(x) = A_1(\tau_1);$$

Case 2. *If* $\tau_1 = x^* = x_2 < \tilde{x}$, *or* $\tau_1 = \tilde{x} = x_2 < x^*$, *then*

$$\lim_{x \to \tau_1} \sqrt{\tau_1 - x}\,\phi_2(x) = A_2(\tau_1);$$

Case 3 *If* $\tau_1 = x_2 < \min\{\tilde{x}, x^*\}$, *then*

$$\lim_{x \to \tau_1} \sqrt{\tau_1 - x}\,\phi_1'(x) = A_3(\tau_1);$$

Case 4 *If* $\tau_1 = x^* = \tilde{x} < x_2$, *then*

$$\lim_{x \to \tau_1} (\tau_1 - x)^2\phi_2(x) = A_4(\tau_1),$$

where A_k *for* $k = 1, 2, 3, 4$ *are constants.*

Four types of tail asymptotics

Theorem

For $t \geq 0$, let $V_2(t) = V_2(A)$ if $A = \{(x_1, 0) : x_1 \leq t\}$. Then, for large t, the boundary distribution V_2 has the following four types of tail asymptotics:

Case 1. *If $\tau_1 = x^* < \min\{\tilde{x}, x_2\}$, or $\tau_1 = \tilde{x} < \min\{x^*, x_2\}$, or $\tau_1 = \tilde{x} = x_2$, then*

$$V_1(t) \sim A_1(\tau_1) e^{-\tau_1 t};$$

Case 2. *If $\tau_1 = x^* = x_2 < \tilde{x}$, or $\tau_1 = \tilde{x} = x_2 < x^*$, then*

$$V_1(t) \sim A_2(\tau_1) t^{-1/2} e^{-\tau_1 t};$$

Case 3 *If $\tau_1 = x_2 < \min\{\tilde{x}, x^*\}$, then*

$$V_1(t) \sim A_3(\tau_1) t^{-3/2} e^{-\tau_1 t};$$

Case 4 *If $\tau_1 = x^* = \tilde{x} < x_2$, then*

$$V_1(t) \sim A_4(\tau_1) t e^{-\tau_1 t},$$

where A_k for $k = 1, 2, 3, 4$ are constants.

Literature on kernel methods

- Originated from combinatorics: Knuth[1969] and later developed as the kernel method by Banderier *et al.*[2002];
- Applied to queueing models: Li and Zhao[2011, 2011b], Li, Takakoli and Zhao[2011], Ye[2012], Zafari[2012]
- Miyazawa and Rolski[2009], Miyazawa[2009], Kobayashi and Miyazawa[2011], Dai and Miyazawa[2011] (similar method)
- BVP: Fayolle and Iasnogorodski[1979], Fayolle, Iasnogorodski and Malyshev[1999], Guillemin, Knessl and van Leeuwaarden[2011], Guillemin and van Leeuwaarden[2011], Guillemin and Simonian[2011]
- Malyshev[1972, 1973], Abate and Whitt[1997], Borovkov and Mogul'skii[2001], Kurkova and Suhov[2003], Bousquet-Mélou[2005], Lieshout and Mandjes[2008] (related)

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Thank You!