Existence and uniqueness of invariant measures for SPDEs with two reflecting walls

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Abstract

We study stochastic partial differential equations with two reflecting walls h^1 and h^2, driven by space-time white noise with non-constant diffusion coefficients under periodic boundary conditions. The existence and uniqueness of invariant measures is established. The strong Feller property is also obtained.

The equation

Consider the following stochastic partial differential equations (SPDEs) with two reflecting walls

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}(x,t) \\ \qquad\qquad +\eta(x,t) - \xi(x,t); \\ u(x,0) = u_0(x) \in C(S^1); \\ h^1(x) \le u(x,t) \le h^2(x), \quad \text{for } (x,t) \in Q. \end{cases} \tag{1}$$

$Q := S^1 \times \mathbb{R}_+$, $S^1 := \mathbb{R}(mod 2\pi)$, or $\{e^{i\theta};\ \theta \in \mathbb{R}\}$ denotes a circular ring and the random field $W(x,t) := W(\{e^{i\theta};\ 0 \le \theta \le x\} \times [0,t])$ is a regular Brownian sheet defined on a filtered probability space $(\Omega, P, \mathcal{F}; \mathcal{F}_t)$. The random measures ξ and η are added to equation (1) to prevent the solution from leaving the interval $[h^1,\ h^2]$.

The equation

We assume that the reflecting walls $h^1(x),\ h^2(x)$ are continuous functions satisfying
(H1) $h^1(x) < h^2(x)$ for $x \in S^1$;
(H2) $\frac{\partial^2 h^i}{\partial x^2} \in L^2(S^1)$, where $\frac{\partial^2}{\partial x^2}$ is interpreted in a distributional sense.
We also assume that the coefficients: $f, \sigma : \mathbb{R} \to \mathbb{R}$ satisfy
(F1) there exists $L > 0$ such that

$$|f(z_1) - f(z_2)| + |\sigma(z_1) - \sigma(z_2)| \leq L|z_1 - z_2|,\ z_1,\ z_2 \in \mathbb{R};$$

The equation

Definition 1. A triplet (u, η, ξ) is a solution to the SPDE (1) if
(i) $u = \{u(x,t); (x,t) \in Q\}$ is a continuous, adapted random field (i.e., $u(x,t)$ is $\mathcal{F}_t$-measurable $\forall\ t \geq 0, x \in S^1$) satisfying $h^1(x) \leq u(x,t) \leq h^2(x)$, a.s;
(ii) $\eta(dx,dt)$ and $\xi(dx,dt)$ are positive and adapted (i.e. $\eta(B)$ and $\xi(B)$ is $\mathcal{F}_t$-measurable if $B \subset S^1 \times [0,t]$) random measures on Q satisfying

$$\eta\big(S^1 \times [0,T]\big) < \infty,\ \xi\big(S^1 \times [0,T]\big) < \infty$$

for $T > 0$;

The equation

(iii) for all $t \geq 0$ and $\phi \in C^\infty(S^1)$ we have

$$
\begin{aligned}
& \big(u(t), \phi\big) - \int_0^t (u(s), \phi'')ds - \int_0^t \big(f(u(s)), \phi\big)ds \\
& - \int_0^t \int_{S^1} \phi(x)\sigma(u(x,s))W(dx, ds) \\
= \ & \big(u_0, \phi(x)\big) + \int_0^t \int_{S^1} \phi(x)\eta(dx, ds) \\
& - \int_0^t \int_{S^1} \phi(x)\xi(dx, ds), \ \ a.s, \qquad (2)
\end{aligned}
$$

where $(,)$ denotes the inner product in $L^2(S^1)$ and $u(t)$ denotes $u(\cdot, t)$;
(iv)

$$\int_Q \big(u(x,t) - h^1(x)\big)\eta(dx, dt) = \int_Q \big(h^2(x) - u(x,t)\big)\xi(dx, dt) = 0.$$

The equation

SPDEs with one reflecting barrier at 0 were first studied by Nualart and Pardoux in [4] when the diffusion coefficient $\sigma(\cdot)$ is a constant. When the diffusion coefficient $\sigma(\cdot)$ is not a constant, Donati-Martin and Pardoux proved in [1]that there exists a minimal solution to the SPDE with reflection using penalized approximations. The uniqueness was left out there. In [1], Xu and Z. proved that there exists a unique solution to SPDEs with reflection for general diffusion coefficients. A large deviation principle was also obtained by Xu and Z. Interesting properties (i.e. integration by parts) were obtained by Zambotti in [2].The existence and uniqueness of the solution of of SPDEs with two reflecting walls was established by Yang and Z. in [3].
Our aim here is to establish the existence and uniqueness of invariant measures, as well as the strong Feller property of fully non-linear SPDEs with two reflecting walls (1).

The equation

For SPDEs without reflection, the existence and uniqueness of invariant measures has been studied by many people, see Sowers [4], Mueller [3], Peszat and Zabczyk [2], Da Prato and Zabczyk [2]. For SPDEs with reflection, when the diffusion coefficient σ is a constant, existence and uniqueness of invariant measures was obtained by Otobe [5], [1]. The strong Feller property of SPDEs has been studied by several authors, see Peszat and Zabczyk [2], Da Prato and Zabczyk [2]. The strong Feller property of SPDEs with reflection at 0 was first proved by Z. in [4].
For the existence of invariant measures, the continuity of the solution with respect to the solutions of some random obstacle problems plays an important role. For the uniqueness, we adopted a coupling method. This method was first used by Mueller [3] for proving the uniqueness of invariant measures for SPDEs (with no reflection). Because of the reflection, we need to establish a kind of uniform coupling for approximating solutions. The strong Feller property of SPDEs with two reflecting walls will be obtained in a

Existence of invariant measures

Denote by $\mathcal{B}(C(S^1))$ the σ-field of all Borel subsets of $C(S^1)$. We denote by $u(x, t, u_0)$ the solution of equation (1) and by $P_t(u_0, \cdot)$ the corresponding transition function

$$P_t(u_0, \Gamma) = P(u(\cdot, t, u_0) \in \Gamma), \ \Gamma \in \mathcal{B}(C(S^1)), \ t > 0,$$

where u_0 is the initial condition.
Theorem 1 Suppose the hypotheses (H1)-(H2), (F1) hold. Then there exists an invariant measure to equation (1) on $C(S^1)$.

Sketch of the proof

According to Krylov-Bogolyubov theorem (see [2]), we aim to show that the family of measures

$$\frac{1}{T}\int_0^T P_t(u_0,\cdot)dt, T \geq 1$$

is tight. Thus, if we show that the family $\{P_t(u_0,\cdot);\ t \geq 1\}$ is tight, then there exists an invariant measure for equation (1). So we need to show that for any $\varepsilon > 0$ there is a compact set $K \subset C(S^1)$ such that

$$P(u(t) \in K) \geq 1-\varepsilon, \quad \text{for any } t \geq 1.$$

where $u(t) = u(t, u_0) = u(\cdot, t, u_0)$.

Sketch of the proof

On the other hand, for any $t \geq 1$, we have by the Markov property

$$P(u(t) \in K) \;=\; \mathbb{E}\big(P_1(u(t-1), K)\big). \tag{3}$$

Thus it is enough to show $P_1(u(t-1), K) \geq 1 - \varepsilon$, for any $t \geq 1$. As $h^1(\cdot) \leq u(t-1)(\cdot) \leq h^2(\cdot)$, it suffices to find a compact subset $K \subset C(S^1)$ such that

$$P_1(g, K) \geq 1 - \varepsilon, \quad \textit{for all } g \in C(S^1) \textit{ with } h^1 \leq g \leq h^2. \tag{4}$$

Sketch of the proof

Put

$$v(x,t,g) = \int_0^t \int_{S^1} G_{t-s}(x,y) f(u(y,s,g)) dyds + \int_0^t \int_{S^1} G_{t-s}(x,y)\sigma(u(y,s,g))W(dy,ds), \quad (5)$$

where $G_t(x,y)$ is the Green's function of the heat equation on S^1. Then u can be written as

$$u(x,t,g) - \int_{S^1} G_t(x,y)g(y)dy = v(x,t,g) + \int_0^t \int_{S^1} G_{t-s}(x,y)\eta(g)(dx,dt) - \int_0^t \int_{S^1} G_{t-s}(x,y)\xi(g)(dx,dt),$$

where $\eta(g)$, $\xi(g)$ indicates the dependence of the random measures on the initial condition g. Put

$$\bar{u}(x,t,g) = u(x,t,g) - \int_{S^1} G_t(x,y)g(y)dy$$

Sketch of the proof

Then $(\bar{u}, \eta, \xi)$ solves a random obstacle problem. In particular, we have (see [3]) the following inequality

$$\|\bar{u}(g) - \bar{u}(\hat{g})\|_\infty^1 \leq 2\|v(g) - v(\hat{g})\|_\infty^1,$$

where $\|\omega\|_\infty^1 := \sup_{x\in S^1, t\in[0,1]} |\omega(x,t)|$. And $\bar{u}$ is a continuous functional of v and denoted by $u = \Phi(v)$, where $\Phi(\cdot): \ C(S^1 \times [0,1]) \to C(S^1 \times [0,1])$ is continuous.

Sketch of the proof

In particular, $\bar{u}(\cdot, 1, g)$ is also a continuous functional of v, from $C(S^1 \times [0,1])$ to $C(S^1)$. We denote this functional by Φ_1, i.e. $\bar{u}(\cdot, 1, g) = \Phi_1(v(\cdot, g))$, where $v(\cdot, g) = v(\cdot, \cdot, g)$. If K'' is a compact subset of $C(S^1 \times [0,1]))$, then $K' = \Phi_1(K'')$ is a compact subset in $C(S^1)$ and

$$\begin{aligned} P(\bar{u}(\cdot, 1, g) \in K') &= P(\bar{u}(\cdot, 1, g) \in \Phi_1(K'')) \\ &\geq P(v(\cdot, g) \in K''). \end{aligned} \tag{6}$$

Next, we want to find a compact set $K''(\subset C(S^1 \times [0,1])$ such that

$$P(v(\cdot, g) \in K'') \geq 1 - \varepsilon, \quad \textit{for all } g \in C(S^1) \textit{ with } h^1 \leq g \leq h^2. \tag{7}$$

This is possible because it can be shown that for $0 < \alpha < \frac{1}{4}$ and $\kappa > 0$, there exists a random variable $Y(g)$ such that with probability one, for all $x, y \in S^1$ and $s, t \in [0,1]$,

Sketch of the proof

$$|v(x,t,g) - v(y,s,g)| \leq Y(g)(d((x,t),(y,s)))^{\alpha-\kappa} \quad (8)$$

$$\mathbb{E}(Y(g))^{\frac{1}{\kappa}} \leq C_0, \quad (9)$$

where $d((x,t),(y,s)) := \left(r^2(x,y) + (t-s)^2\right)^{\frac{1}{2}}$ with $r(x,y)$ the length of the shortest arc of S^1 connecting x with y and C_0 is independent of g.

For $r > 0$, $K_r := \{v; \|v\|_\alpha \leq r\}$ is a compact subset of $C(S^1 \times [0,1])$. In view of (8), we see that for given $\varepsilon > 0$, there exists r_0 such that

$$P(v(\cdot, g) \in K_{r_0}^c) \leq \varepsilon, \quad \textit{for all } g \textit{ with } h^1 \leq g \leq h^2.$$

Choosing $K'' = K_{r_0}$, we obtain (7). Hence $P(\bar{u}(\cdot, 1, g) \in K') \geq 1 - \varepsilon$ for all $g \in C(S^1)$ with $h^1 \leq g \leq h^2$.

Sketch of the proof

On the other hand, it is easy to see that there is a compact subset $K_0 \subset C(S^1)$ such that

$$\{\int_{S^1} G_1(x,y)g(y)dy; \quad h^1 \leq g \leq h^2\} \subset K_0$$

Define $K = K' + K_0$. We have

$$P_1(g,K) = P(u(\cdot,1,g) \in K) \geq P(\bar{u}(\cdot,1,g) \in K') \geq 1-\varepsilon,$$

for all $g \in C(S^1)$ with $h^1 \leq g \leq h^2$. This finishes the proof.

Uniqueness of invariant measures

For the uniqueness of invariant measures, we need the following proposition. Put $u(x,t) = u(x,t,u_0)$.

Proposition 2. Under the assumption in Theorem 1, for any $p \geq 1$, $T > 0$, $\sup\limits_{\varepsilon,\delta} \mathbb{E}(\|u^{\varepsilon,\delta}\|_\infty^T)^p < \infty$ and $u^{\varepsilon,\delta}$ converges uniformly on $S^1 \times [0,T]$ to u as $\varepsilon, \delta \to 0$ a.s, where u, $u^{\varepsilon,\delta}$ are the solutions of equation (1) and the penalized SPDEs

$$\begin{cases} \frac{\partial u^{\varepsilon,\delta}(x,t)}{\partial t} = \frac{\partial^2 u^{\varepsilon,\delta}(x,t)}{\partial x^2} + f(u^{\varepsilon,\delta}(x,t)) + \sigma(u^{\varepsilon,\delta}(x,t))\dot{W}(x,t) \\ \qquad\qquad + \frac{1}{\delta}(u^{\varepsilon,\delta}(x,t) - h^1(x))^- - \frac{1}{\varepsilon}(u^{\varepsilon,\delta}(x,t) - h^2(x))^+; \\ u^{\varepsilon,\delta}(x,0) = u_0(x). \end{cases}$$

The following result is the uniqueness of invariant measures.
Theorem 3. Under the assumptions in Theorem 1 and that $\sigma \geq L_0$ for some constant $L_0 > 0$, there is a unique invariant measure for the equation (1).

Sketch of the proof. It is not obvious how to construct a successful coupling directly on the SPDEs with reflection. Instead, we will construct the coupling to approximating equations and then try to pass to the limit. For given two initial functions $u^1(x,0)$ and $u^2(x,0)$, we aim to construct two coupled processes $u^1(x,t)$, $u^2(x,t)$ satisfying equation (1), driven by different white noises on a probability space $(\Omega, \mathcal{F}, P)$, such that

$$\lim_{t\to\infty} P\big(\sup_{x\in S^1} |u^1(x,t) - u^2(x,t)| \neq 0\big) = 0. \tag{10}$$

Sketch of the proof of uniqueness

We first assume $u^1(x,0) \geq u^2(x,0)$, $x \in S^1$. We want to construct two independent space-time white noises $W_1(x,t)$, $W_2(x,t)$ defined on some probability space $(\Omega, \mathcal{F}, P)$, and a solution u, v of the following SPDEs with two reflecting walls

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}_1(x,t) \\
&\quad + \eta_1(x,t) - \xi_1(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + f\big(v(x,t)\big) + \eta_2(x,t) - \xi_2(x,t) \\
&\quad + \sigma\big(v(x,t)\big)\Big[(1-|u-v|\wedge 1)^{\frac{1}{2}}\dot{W}_1(x,t) \\
&\qquad + (|u-v|\wedge 1)^{\frac{1}{2}}\dot{W}_2(x,t)\Big], \\
u(x,0) &= u^1(x,0), \quad v(x,0) = u^2(x,0).
\end{aligned} \tag{11}
$$

Note that the coefficients in the second equation in (11) is not Lipschitz. The existence of a solution of equation (11) is not immediate.

Sketch of the proof of uniqueness

We will give a construction of a solution on some probability space. The construction will also be used to prove the successful coupling

$$\lim_{t\to\infty} P\Big(\sup_{x\in S^1} |u(x,t) - v(x,t)| \neq 0\Big) = 0.$$

For $0 \le z \le 1$, set

$$\begin{aligned} f_n(z) &= \big(z + \frac{1}{n}\big)^{\frac{1}{2}} - \big(\frac{1}{n}\big)^{\frac{1}{2}}, \\ g_n(x) &= \big(1 - f_n(z)^2\big)^{\frac{1}{2}}. \end{aligned}$$

We have $f_n(z)^2 + g_n(z)^2 = 1$ and that $f_n(z) \to z^{\frac{1}{2}}$, $g_n(z) \to (1-z)^{\frac{1}{2}}$ uniformly as $n \to \infty$, for $z \in S^1$.
Let $\dot{\overline{W}}_1(x,t)$, $\dot{\overline{W}}_2(x,t)$ be two independent space-time white noises defined on a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. Let $\overline{u}, \overline{v}^n$ be the unique solution of the following SPDEs with two reflecting walls

Sketch of the proof of uniqueness

$$\begin{aligned}\frac{\partial \overline{u}(x,t)}{\partial t} &= \frac{\partial^2 \overline{u}(x,t)}{\partial x^2} + f\left(\overline{u}(x,t)\right) + \sigma\left(\overline{u}(x,t)\right)\dot{\overline{W}}_1(x,t) \\ &\quad + \overline{\eta}_1(x,t) - \overline{\xi}_1(x,t), \qquad (12)\end{aligned}$$

Sketch of the proof of uniqueness

$$
\begin{aligned}
\frac{\partial \overline{v}^n(x,t)}{\partial t} &= \frac{\partial^2 \overline{v}^n(x,t)}{\partial x^2} + f\left(\overline{v}^n(x,t)\right) + \bar{\eta}_2^n(x,t) - \bar{\xi}_2^n(x,t) \\
&\quad + \sigma\left(\overline{v}^n(x,t)\right)\left[g_n(|\bar{u} - \overline{v}^n| \wedge 1)\dot{\overline{W}}_1(x,t)\right. \\
&\qquad \left. + f_n(|\overline{u} - \overline{v}^n| \wedge 1)\dot{\overline{W}}_2(x,t)\right], \\
\overline{u}(x,0) &= u^1(x,0), \quad \overline{v}(x,0) = u^2(x,0). \qquad (13)
\end{aligned}
$$

We can show that the vector $(\overline{u}, \overline{v}^n, \overline{W}_1, \overline{W}_2)$ is tight. By Skorohod's representation theorem, there exist random fields (u, v^n, W_1, W_2), $n \geq 1$ on some probability space $(\Omega, \mathcal{F}, P)$ such that (u, v^n, W_1, W_2) has the same law as $(\overline{u}, \overline{v}^n, \overline{W}_1, \overline{W}_2)$ and that the following SPDEs with two reflecting walls hold

Sketch of the proof of uniqueness

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\left(u(x,t)\right) + \sigma\left(u(x,t)\right)\dot{W}_1(x,t) \\
&\quad +\eta_1(x,t) - \xi_1(x,t), \\
\frac{\partial v^n(x,t)}{\partial t} &= \frac{\partial^2 v^n(x,t)}{\partial x^2} + f\left(v^n(x,t)\right) + \eta_2^n(x,t) - \xi_2^n(x,t) \\
&\quad +\sigma\left(v^n(x,t)\right)\left[g_n(|u-v^n|\wedge 1)\dot{W}_1(x,t)\right. \\
&\qquad \left. +f_n(|u-v^n|\wedge 1)\dot{W}_2(x,t)\right], \\
u(x,0) &= u^1(x,0), \quad v^n(x,0) = u^2(x,0). \qquad (14)
\end{aligned}
$$

Furthermore, $v^n \to v$ uniformly almost surely as $n \to \infty$.

Sketch of proof of uniqueness

We can prove that the limit (u, v) satisfies the following SPDEs with two reflecting walls

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}_1(x,t) \\
&\quad +\eta_1(x,t) - \xi_1(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + f\big(v(x,t)\big) + \eta_2(x,t) - \xi_2(x,t) \\
&\quad +\sigma\big(v(x,t)\big)\Big[(1-|u-v|\wedge 1)^{\frac{1}{2}}\dot{W}_1(x,t) \\
&\qquad +(|u-v|\wedge 1)^{\frac{1}{2}}\dot{W}_2(x,t)\Big], \\
u(x,0) &= u^1(x,0), \quad v(x,0) = u^2(x,0). \qquad (15)
\end{aligned}
$$

The next step is to show that u, v admits a successful coupling. To this end, consider the following approximating SPDEs

Sketch of proof of uniqueness

$$
\begin{cases}
\frac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{\partial^2 u^{\varepsilon,\delta}}{\partial x^2} + f(u^{\varepsilon,\delta}) + \frac{1}{\delta}(u^{\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(u^{\varepsilon,\delta} - h^2)^+ \\
+\sigma(u^{\varepsilon,\delta})\dot{W}_1; \\
\frac{\partial v^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 v^{n,\varepsilon,\delta}}{\partial x^2} + f(v^{n,\varepsilon,\delta}) + \frac{1}{\delta}(v^{n,\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(v^{n,\varepsilon,\delta} - h^2)^+ \\
\qquad +\sigma(v^{n,\varepsilon,\delta})\big[g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_1(x,t) \\
\qquad +f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_2(x,t)\big]; \\
u^{\varepsilon,\delta}(x,0) = u^1(x,0),\ v^{n,\varepsilon,\delta}(x,0) = u^2(x,0).
\end{cases}
\tag{16}
$$

We may and will assume that $f(u)$ is non-increasing. Otherwise, we consider $\tilde{u} := e^{-Lt}u$, $\tilde{v} := e^{-Lt}v$, where L is the Lipschitz constant in (F1). Apply Proposition 2 to conclude that $u^{\varepsilon,\delta}(x,t) \to u(x,t)$, $v^{n,\varepsilon,\delta}(x,t) \to v^n(x,t)$ uniformly on $S^1 \times [0, T]$ (for any $T > 0$) as $\varepsilon, \delta \to 0$.

Sketch of proof of uniqueness

Let

$$U^{n,\varepsilon,\delta}(t) = \int_{S^1} (u^{\varepsilon,\delta}(x,t) - v^{n,\varepsilon,\delta}(x,t))dx. \tag{17}$$

It follows from the above equation that

$$U^{n,\varepsilon,\delta}(t) = \int_{S^1} (u_1(x,0) - u_2(x,0))dx + \int_0^t C^{n,\varepsilon,\delta}(s)ds + M^{n,\varepsilon,\delta}(t), \tag{18}$$

where

$$\begin{aligned} C^{n,\varepsilon,\delta}(t) &= \int_{S^1} \Big\{ f(u^{\varepsilon,\delta}) - f(v^{n,\varepsilon,\delta}) + \frac{1}{\delta}(u^{\varepsilon,\delta} - h^1)^-(x,t) \\ &\quad - \frac{1}{\delta}(v^{n,\varepsilon,\delta} - h^1)^-(x,t) \\ &\quad - \big(\frac{1}{\varepsilon}(u^{\varepsilon,\delta} - h^2)^+(x,t) - \frac{1}{\varepsilon}(v^{n,\varepsilon,\delta} - h^2)^+(x,t)\big) \Big\} dx \\ &\leq 0, \end{aligned}$$

Sketch of proof of uniqueness

$$
\begin{aligned}
M^{n,\varepsilon,\delta}(t) \;=\; & \int_0^t \int_{S^1} \sigma(u^{\varepsilon,\delta}(x,s)) W_1(dx,ds) \\
& - \int_0^t \int_{S^1} \sigma(v^{n,\varepsilon,\delta}(x,s)) g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1) \dot{W}_1(dx,ds) \\
& - \int_0^t \int_{S^1} \sigma(v^{n,\varepsilon,\delta}(x,s)) f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1) \dot{W}_2(dx,ds).
\end{aligned}
$$

Sketch of proof of uniqueness

Observe that

$$\begin{aligned} & \lim_{\varepsilon,\delta\to 0} U^{n,\varepsilon,\delta}(t) \\ = \quad & U^n(t) := \int_{S^1} (u(x,t) - v^n(x,t))dx, \end{aligned} \tag{19}$$

and

$$\begin{aligned} & \lim_{\varepsilon,\delta\to 0} M^{n,\varepsilon,\delta}(t) \\ = \quad & M^n(t) := \int_0^t \int_{S^1} \sigma(u(x,s)) W_1(dx,ds) \\ & - \int_0^t \int_{S^1} \sigma(v^n(x,s)) g_n(|u - v^n| \wedge 1) \dot{W}_1(dx,ds) \\ & - \int_0^t \int_{S^1} \sigma(v^n(x,s)) f_n(|u - v^n| \wedge 1) \dot{W}_2(dx,ds). \end{aligned} \tag{20}$$

Sketch of proof of uniqueness

Letting $\varepsilon, \delta \to 0$ in (18) we see that

$$U^n(t) = \int_{S^1} (u_1(x,0) - u_2(x,0))dx + A^n(t) + M^n(t), \qquad (21)$$

where $A^n(t) = \lim_{\varepsilon,\delta\to 0} \int_0^t C^{n,\varepsilon,\delta}(s)ds$ is a continuous, adapted non-increasing process.

Sketch of proof of uniqueness

Now, sending n to ∞ we obtain

$$U(t) = \int_{S^1} (u_1(x,0) - u_2(x,0))dx + A(t) + M(t), \qquad (22)$$

where

$$U(t) = \int_{S^1} (u(x,t) - v(x,t))dx,$$

$$\begin{aligned} M(t) &= \int_0^t \int_{S^1} \sigma(u(x,s)) W_1(dx,ds) \\ &\quad - \int_0^t \int_{S^1} \sigma(v(x,s))(1 - |u-v| \wedge 1)^{\frac{1}{2}} \dot{W}_1(dx,ds) \\ &\quad - \int_0^t \int_{S^1} \sigma(v(x,s))(|u-v| \wedge 1)^{\frac{1}{2}} \dot{W}_2(dx,ds), \end{aligned}$$

and $A(t) = \lim_{n\to\infty} A^n(t)$ a continuous, adapted non-increasing process. The existence of the limits of A^n follows from the existence of the limit of U^n and M^n.

Sketch of proof of uniqueness

In view of the assumption on σ and the boundedness of the walls h^1, h^2, it is easy to verify that

$$\frac{d < M > (t)}{dt} \geq C_0 U(t) \tag{23}$$

for some positive constant C_0. Thus, there exists a non-negative adapted process $V(t)$ such that

$$\frac{d < M > (t)}{dt} = U(t)V(t), \qquad V(t) \geq C_0.$$

Let

$$\begin{aligned} \phi(t) &= \int_0^t V(s)ds, \\ X(t) &= U(\phi^{-1}(t)). \end{aligned} \tag{24}$$

Then the time-changed process X satisfies the following equation

$$X(t) = U(0) + \tilde{A}(t) + \int_0^t X^{\frac{1}{2}}(s)dB(s), \tag{25}$$

Sketch of proof of uniqueness

Let $Y(t) = 2X^{\frac{1}{2}}(t)$. We have

$$Y(t) = Y(0) + 2\int_0^t \frac{1}{Y(s)} d\tilde{A}(s) - \frac{1}{2}\int_0^t \frac{1}{Y(s)} ds + B(t). \quad (26)$$

As $\tilde{A}$ is non-increasing, it follows that

$$0 \leq Y(t) \leq Y(0) + B(t). \quad (27)$$

The property of one dimensional Brownian motion implies that Y hits 0 with probability 1. Hence

$$\lim_{t\to\infty} P\Big(\sup_{x\in S^1} |u(x,t) - v(x,t)| \neq 0\Big) = 0.$$

Strong Feller property

Let $H = L^2(S^1)$. If $\varphi \in B_b(H)$, we define, for $0 \le t \le T$,

$$P_t\varphi(g) = \mathbb{E}\varphi(u(x,t,g)).$$

Theorem 4. Under the hypotheses (H1)-(H2), (F1)-(F2) and that $p_1 \le |\sigma(\cdot)| \le p_2$ for some constants $p_1,\ p_2 > 0$, then for any $T > 0$ there exists a constant C'_T such that for all $\varphi \in B_b(H)$ and $t \in (0, T]$,

$$|P_t\varphi(u_0^1) - P_t\varphi(u_0^2)| \ \le \ \frac{C'_T}{\sqrt{t}}\|\varphi\|_\infty |u_0^1 - u_0^2|_H, \qquad (28)$$

for $u_0^1,\ u_0^2 \in H$ with $h^1(x) \le u_0^1(x),\ u_0^2(x) \le h^2(x)$, where $\|\varphi\|_\infty = \sup_{u_0}|\varphi(u_0)|$. In particular, $P_t,\ t > 0$, is strong Feller.

Ideas of the proof

We again consider approximating equations of the reflected SPDEs. We use Elworthy-Li formula to obtain a uniform strong Feller property for the approximating equations. The result follows by passing to the limit.

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