Switching Diffusion Processes

G. Yin

Wayne State University, gyin@math.wayne.edu

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This talk reports some of recent findings involving joint work with

C. Zhu (Univ. Wisconsin, Milwaukee)

R.Z. Khasminskii (WSU)

Q. Song (Hong Kong City Univ.)

F. Wu (Huazhong Univ. of Tech.)

G. Zhao (WSU)

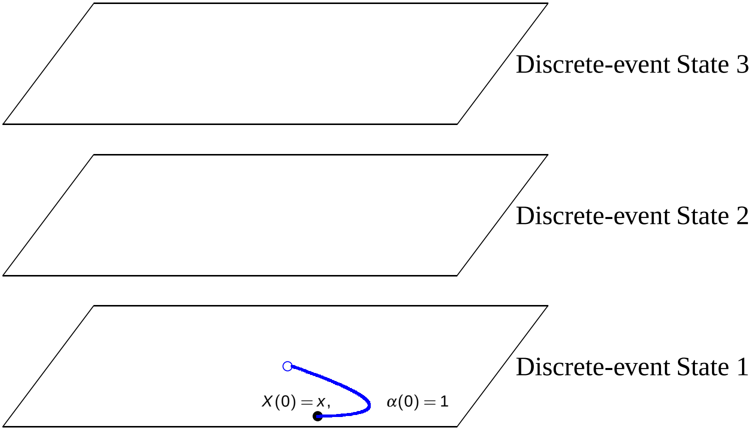
Other coauthors of related work X. Mao (Univ. of Strathclyde), C. Yuan (Univ. of Swansea), F. Xi (Beijing Inst. Tech.)

Outline

1. Switching Diffusions: Formulation & Review of Some Recent Results
   - Formulation
   - Motivational Examples
   - Recurrence
   - Ergodicity
2. Seemingly Not Much Different from a Diffusion?
3. Explosion Suppression & Stabilization
   - Explosion Suppression Using Feedback Control
   - Stabilization
4. Numerical Solutions for Controlled Switching Diffusions

Switching Diffusions: Formulation & Review of Some Recent Results

Switching Diffusion: An Illustration



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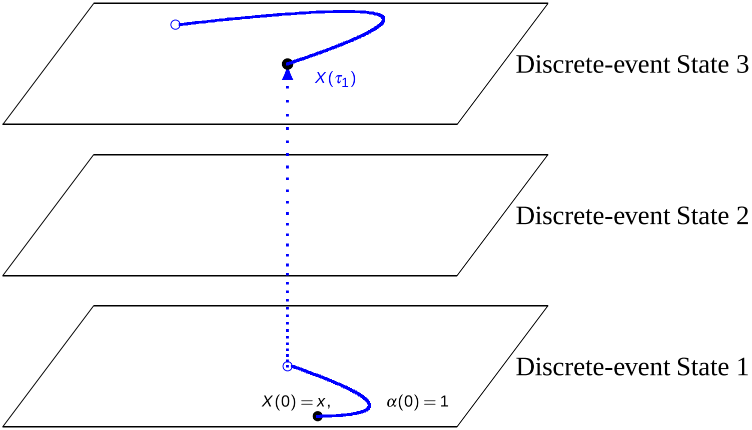

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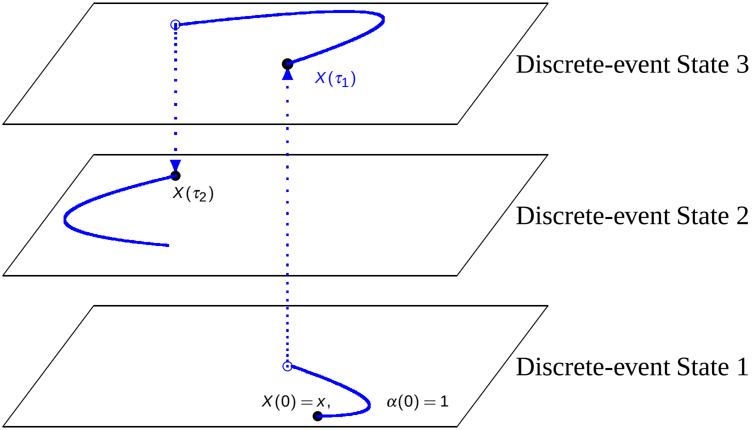

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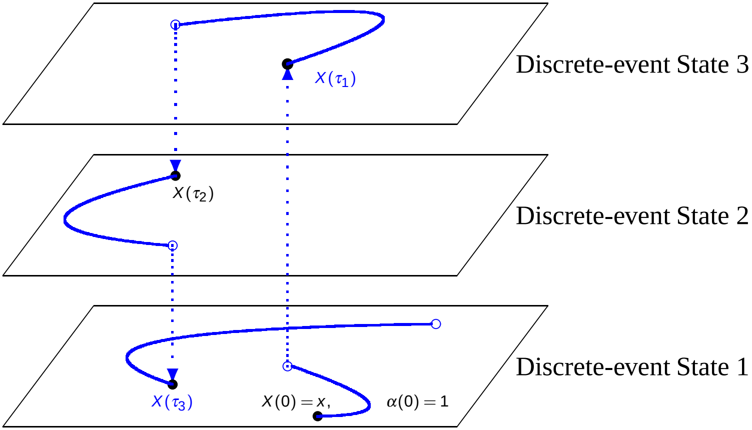

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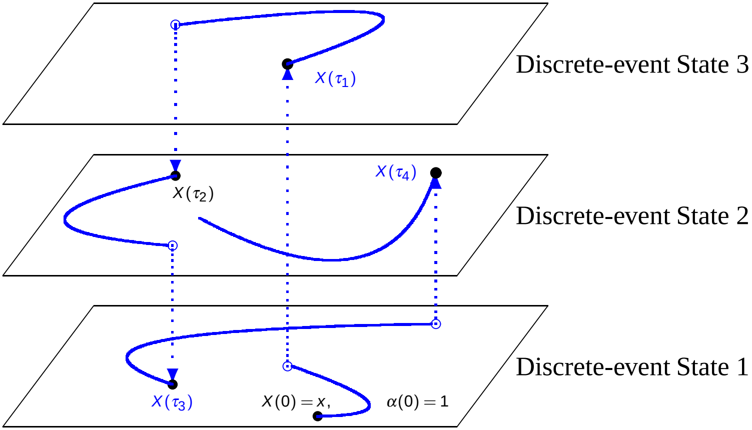

Figure: A “Sample Path” of the Switching Diffusion $(X(t), \alpha(t))$.

Main Features

- continuous dynamics & discrete events coexist
- switching is used to model random environment or other random factors that cannot be formulated by the usual differential equations
- problems naturally arise in applications such as distributed, cooperative, and non-cooperative games, wireless communication, target tracking, reconfigurable sensor deployment, autonomous decision making, learning, etc.
- traditional ODE or SDE models are no longer adequate
- non-Gaussian distribution

Switching Diffusions

$\mathscr{M} = \{1, \ldots, m\}$
$\alpha(\cdot)$: taking values in $\mathscr{M}$.
$w(t)$: d-dimensional standard Brownian motion
$b(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r$
$\sigma(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r \times \mathbb{R}^d$

$$\begin{aligned} &dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t), \\ &X(0) = x, \ \alpha(0) = \alpha, \end{aligned} \tag{1}$$

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$$\begin{aligned} &P\{\alpha(t+\Delta) = j | \alpha(t) = i, (X(s), \alpha(s)), s \leq t\} \\ &\quad = q_{ij}(X(t))\Delta + o(\Delta),\ i \neq j. \end{aligned} \tag{2}$$

Formulation (cont.)

$Q(x) = (q_{ij}(x))$: generator associated with $\alpha(t)$ satisfying

$$q_{ij}(x) \geq 0, \text{ if } j \neq i, \text{ and } \sum_{j=1}^{m} q_{ij}(x) = 0, \quad i = 1, 2, \ldots, m$$

$\mathscr{L}$: generator of $(X(t), \alpha(t))$. For each $i \in \mathscr{M}$, and any $g(\cdot, i) \in C^2(\mathbb{R}^r)$,

$$\mathscr{L}g(x,i) = \frac{1}{2}\mathrm{tr}(a(x,i)\nabla^2 g(x,i)) + b'(x,i)\nabla g(x,i) + Q(x)g(x,\cdot)(i) \qquad (3)$$

where

$\nabla g(\cdot, i)$ & $\nabla^2 g(\cdot, i)$: gradient & Hessian of $g(\cdot, i)$,

$a(x,i) = \sigma(x,i)\sigma'(x,i)$,

$Q(x)g(x,\cdot)(i) = \sum_{j=1}^{m} q_{ij}(x) g(x,j)$.

Main Difficulty of x-Dependnet Switching

- Consider $(X(t), \alpha(t))$ with two different initial data $(X(0), \alpha(0)) = (x, \alpha)$ & $(X(0), \alpha(0)) = (y, \alpha)$, $y \neq x$.

- Since $Q(x)$ depends on x, $\alpha^{x,\alpha}(t) \neq \alpha^{y,\alpha}(t)$ infinitely often even though $\alpha^{x,\alpha}(0) = \alpha^{y,\alpha}(0) = \alpha$.

Stock Price Models

- Variables and parameters
  - $S(t)$: stock price
  - $w(\cdot)$: stand Brownian motion
  - μ: return (appreciation) rate
  - σ: volatility
- Traditional GBM model:

$$dS(t) = \mu S(t)dt + \sigma S(t)dw.$$

- Regime-switching market model:

$$dS(t) = \mu(\alpha(t))S(t)dt + \sigma(\alpha(t))S(t)dw.$$

  - $\alpha(\cdot)$ continuous-time Markov chain independent of $w(\cdot)$
  - $\alpha(\cdot)$: market mode, investor's mode, & other economic factors (e.g., bull, bear)

Consensus Problems: Schooling (Couzin, [Nature, 2005])

Mean-Field Model

- Originated from statistical mechanics, mean-field models are concerned with many-body systems. To overcome the difficulty of interactions due to the many bodies, one of the main ideas is to replace all interactions to any one body with an average or effective interaction.
- $\alpha(t)$: with $\mathscr{M} = \{1, 2, \dots, m_0\}$.
- Consider an ℓ-body mean-field model For $i = 1, 2, \dots, \ell$,

$$
\begin{aligned}
dX_i(t) = &\ \big[\gamma(\alpha(t))X_i(t) - X_i^3(t) - \beta(\alpha(t))(X_i(t) - \overline{X}(t))\big]dt \\
&+\sigma_{ii}(X(t), \alpha(t))dw_i(t), \\
\overline{X}(t) = &\ \frac{1}{\ell}\sum_{j=1}^{\ell} X_j(t), \\
X(t) = &\ (X_1(t), X_2(t), \dots, X_\ell(t))',
\end{aligned}
\tag{4}
$$

$\gamma(i) > 0$ and $\beta(i) > 0$ for $i \in \mathscr{M}$.

Regularity

Definition

Regularity. A Markov process $Y^{x,\alpha}(t) = (X^{x,\alpha}(t), \alpha^{x,\alpha}(t))$ is said to be *regular*, if for any $0 < T < \infty$,

$$\mathbf{P}\{\sup_{0\le t\le T} |X^{x,\alpha}(t)| = \infty\} = 0. \tag{5}$$

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Remark

Let $\beta_n := \inf\{t : |X^{x,\alpha}(t)| = n\}$. Then $\{\beta_n\}$ is monotonically increasing and hence has a (finite or infinite) limit. It follows that the process is regular iff

$$\beta_n \to \infty \text{ almost surely as } n \to \infty. \tag{6}$$

Recurrence

Definition

(i) *Recurrence.* For $U := D \times J$, where $J \subset \mathcal{M}$ and $D \subset \mathbb{R}^r$ is an open set with compact closure, let $\sigma_U^{x,\alpha} = \inf\{t : Y^{x,\alpha}(t) \in U\}$. A regular process $Y^{x,\alpha}(\cdot)$ is *recurrent* w.r.t. U if

$$\mathbf{P}\{\sigma_U^{x,\alpha} < \infty\} = 1 \text{ for any } (x,\alpha) \in D^c \times \mathcal{M}.$$

(ii) *Positive and Null Recurrence.* A recurrent process satisfying $\mathbf{E}\sigma_U^{x,\alpha} < \infty$ is said to be *positive recurrent* w.r.t. U; otherwise, the process is *null recurrent* w.r.t. U.

Recurrence Is Independent of Sets

(i) The process $(X(t), \alpha(t))$ is (positive) recurrent w.r.t. $D \times \mathcal{M}$ if and only if it is (positive) recurrent w.r.t. $D \times \{\ell\}$, where $D \subset \mathbb{R}^r$ is a bounded open set with compact closure and $\ell \in \mathcal{M}$.

(ii) If the process $(X(t), \alpha(t))$ is (positive) recurrent w.r.t. some $U = D \times \mathcal{M}$, where $D \subset \mathbb{R}^r$, then it is (positive) recurrent w.r.t. $\widetilde{U} = \widetilde{D} \times \mathcal{M}$, where $\widetilde{D} \subset \mathbb{R}^r$ is any nonempty open set.

Positive Recurrence

Theorem

A necessary and sufficient condition for positive recurrence with respect to a domain $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathscr{M}$ *is: For each* $i \in \mathscr{M}$*, there exists a nonnegative function* $V(\cdot, i) : D^c \mapsto \mathbb{R}$ *s.t.* $V(\cdot, i)$ *is twice continuously differentiable and that*

$$\mathscr{L}V(x,i) = -1, \ \ (x,i) \in D^c \times \mathscr{M}. \tag{7}$$

Let $u(x,i) = \mathbf{E}_{x,i}\sigma_D$. *It is the* *smallest positive sol'n* *to*

$$\begin{cases} \mathscr{L}u(x,i) = -1, & (x,i) \in D^c \times \mathscr{M}, \\ u(x,i) = 0, & (x,i) \in \partial D \times \mathscr{M}. \end{cases} \tag{8}$$

Ergodicity

Theorem

A positive recurrent process $(X(t),\alpha(t))$*has a unique stationary distribution* $\widehat{\nu}(\cdot,\cdot) = (\widehat{\nu}(\cdot,i) : i \in \mathscr{M})$.

Ergodicity

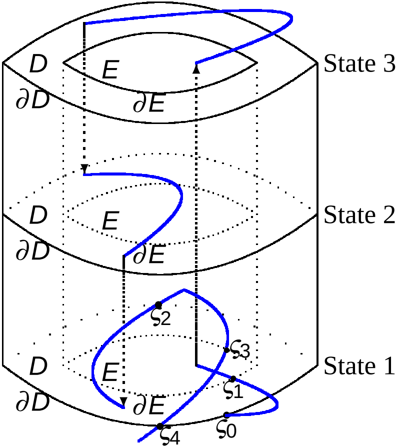


Figure 2: Cycles of $Y(t) = (X(t), \alpha(t))$; $m = 3$ & $\ell = 1$

Seemingly Not Much Different from a Diffusion?

An Example

Consider

$$\dot{x}(t) = A(\alpha(t))x(t) \tag{9}$$

where $\alpha(t)$ has two states $\{1, 2\}$,

$$A(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

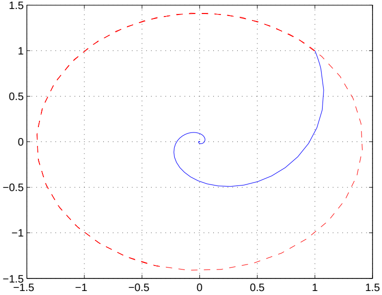
Associated with the hybrid system, there are two ODEs

$$\dot{x}(t) = A(1)x(t), \quad \text{and} \tag{10}$$

$$\dot{x}(t) = A(2)x(t) \tag{11}$$

switching back and forth according to $\alpha(t)$.

Phase Portrait of the Components



Phase portraits of the 'component' with a center (in dashed line) and the 'component' with a stable node (in solid line) with the same initial condition $x_0 = [1, 1]'$

Phase Portrait of Hybrid System

The phase portrait is given below.

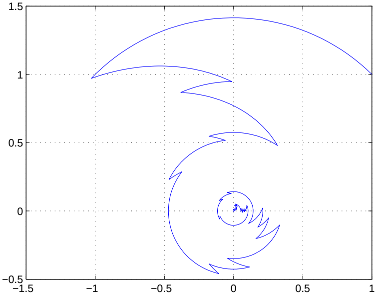


Figure: Switching linear system: Phase portrait of (9) with $x_0 = [1, 1]'$.

Seemingly Not Much Different from Diffusions without Switching?

Q: When we have a coupled system with $\mathscr{M} = \{1,2\}$ and two stable linear systems, do we always get a stable system?

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Q: When we have a coupled system with $\mathscr{M} = \{1, 2\}$ and two stable linear systems, do we always get a stable system?

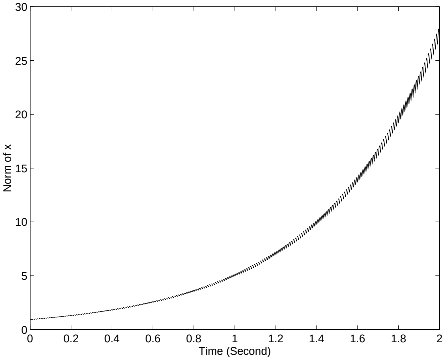
Consider $\dot{x} = A(\alpha(t))x + B(\alpha(t))u(t)$, and a state feedback $u(t) = K(\alpha(t))x(t)$. Then one gets

$$\dot{x} = [A(\alpha(t)) - B(\alpha(t))K(\alpha(t))]x.$$

Suppose that $\alpha(t) \in \{1, 2\}$ such that

$$A(1) - B(1)K(1) = \begin{bmatrix} -100 & 20 \\ 200 & -100 \end{bmatrix}, \ A(2) - B(2)K(2) = \begin{bmatrix} -100 & 200 \\ 20 & -100 \end{bmatrix}.$$

The hybrid system is unstable



[L.Y. Wang, P.P. Khargonecker, and A. Beydoun, 1999, deterministic switching system]

Why is the system unstable?

$$\frac{1}{2}[A(1) - B(1)K(1) + A(2) - B(2)K(2)] = \frac{1}{2}\begin{bmatrix} -200 & 220 \\ 220 & -200 \end{bmatrix}$$

is an unstable matrix.

The averaging effect dominates the dynamics.

- Consider a system

$$\dot{x}^{\varepsilon}(t) = b(x^{\varepsilon}(t), \alpha^{\varepsilon}(t)), \ \alpha^{\varepsilon}(t) \sim Q/\varepsilon \tag{12}$$

- each $\dot{x}(t) = b(x(t), i), \ i \in \mathscr{M}$ is stable.
- Q irreducible
- $x^{\varepsilon}(\cdot) \Rightarrow x(\cdot)$ such that

$$\dot{x}(t) = \overline{b}(x(t)), \ \overline{b}(x) = \sum_{i \in \mathscr{M}} \nu_i b(x, i). \tag{13}$$

- System (13) is unstable.
- Use perturbed Liapunov function to show that (12) is unstable.

Explosion Suppression & Stabilization

Regularity: Criterion

Theorem

Suppose that $b(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^r$ *and that* $\sigma(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^{r\times d}$*,*

$$\begin{aligned} & dX(t)=b(X(t),\alpha(t))dt+\sigma(X(t),\alpha(t))dw(t),\ (X(0),\alpha(0))=(x,\alpha),\\ & P\{\alpha(t+\delta)=j|\alpha(t)=i,X(s),\alpha(s),s\le t\}=q_{ij}(X(t))\delta+o(\delta),\ i\neq j. \end{aligned} \tag{14}$$

Suppose that for each $i\in\mathcal{M}$*, both* $b(\cdot,i)$ *and* $\sigma(\cdot,i)$ *are local linear growth and local Lipschitzian and that* $\exists$ *a nonnegative* $V(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^+$ *that is* C^2 *in* $x\in\mathbb{R}^r$ *for each* $i\in\mathcal{M}$ *s.t.* $\exists\gamma_0>0$

$$\begin{aligned} & \mathcal{L}V(x,i)\le\gamma_0V(x,i),\ \textit{for all}\ (x,i)\in\mathbb{R}^r\times\mathcal{M},\\ & V_R:=\inf_{|x|\ge R,\ i\in\mathcal{M}}V(x,i)\to\infty\ \textit{ as }\ R\to\infty. \end{aligned} \tag{15}$$

Then the process $(X(t),\alpha(t))$ *is regular.*

Explosion Suppression

$x \in \mathbb{R}^r$

$f(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r$

$\alpha(t) \in \mathscr{M} = \{1,\ldots,m\}$

$$\frac{dX(t)}{dt} = f(X(t),\alpha(t)) \tag{16}$$

$f(\cdot,i)$ continuous but the growth rate is faster than linear

We wish to stabilize (16).

Motivational Example

- Consider an even simpler problem: the logistic system

$$\dot{x}(t) = x(t)(1 + x(t)), \; x(0) = 1.$$

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  Two things are needed:

  1) extend to a global solution;
  2) stabilization.

Stabilization: What have been done?

- Khasminskii's book (1981): stabilize 2-d system with two white noise
- Arnold (1972): $\dot{x} = Ax$ can can be stabilized by zero mean stationary process iff $\text{tr}(A) < 0$
- Mao (1994) established a general stabilization results of Brownian noise under linear growth condition.
- Wu & Hu (2009) treated one-sided growth condition
- Mao, Yin, and Yuan (2007): showed that both Brownian motion and Markov Chain can be used to stabilize systems.

Motivation (diffusion case)

$$dx = \mu x dt + \sigma x dw, \; x(0) = x_0.$$

$$x(t) = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)\right).$$

$$\text{when } \; \sigma^2 > 2\mu,$$

$$\limsup_t \frac{\log|x(t)|}{t} \leq \left(\mu - \frac{\sigma^2}{2}\right) < 0.$$

This implies exponential stability.

How to Get a Global Solution? Stablization?

- add a diffusion perturbation

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))|X(t)|^{\beta} X(t) dw_1(t)$$

  such that $2\beta - \beta_1 > 0$, where $w_1(\cdot)$ is scalar Brownian motion.

- add another diffusion to get stability

$$\begin{aligned} dX(t) &= f(X(t), \alpha(t))dt + a_1(\alpha(t))|X(t)|^{\beta} X(t) dw_1(t) \\ &\quad + a_2(\alpha(t)) X(t) dw_2(t), \end{aligned} \tag{17}$$

  where $w_2(\cdot)$ is a scalar Brownian motion independent of $w_1(\cdot)$.

Results

- General case:

$$dX(t) = f(X(t), \alpha(t))dt + \sigma_1(X(t), \alpha(t))dw_1 + \sigma_2(X(t), \alpha(t))dw_2. \tag{18}$$

- With proper choice of the perturbations, we get a global solution
- $\limsup_{t\to\infty} P(|X(t)| \geq K_\delta) \leq \delta$
- The resulting system is stable w.p.1. In fact, $\limsup_t \log |X(t)|/t < 0$ w.p.1.

(A) For each $i \in \mathscr{M}$, $f(\cdot,i)$, $\sigma_1(\cdot,i)$, and $\sigma_2(\cdot,i)$ are locally Lipschitz continuous such that

(a) $f(0,i)=0$;

(b) $f'(x,i)x \le K_0(i)(|x|^{\beta_1+2}+|x|^2)$ for each $i \in \mathscr{M}$ and some $\beta_1 > 0$.

(c) for some $\beta > 0$ satisfying $2\beta - \beta_1 > 0$ and some $K_j(i) > 0$ with $j = 1,\ldots,4$ satisfying $2K_1(i) > K_2(i)$ and for each $x \in \mathbb{R}^r$,

$$\begin{aligned} &K_1(i)(|x|^{4+2\beta}-|x|^4) \le \mathrm{tr}(\sigma_1(x,i)\sigma_1'(x,i)xx') \le K_5(i)|x|^{4+2\beta} \\ &\mathrm{tr}(\sigma_1(x,i)\sigma_1'(x,i)) \le K_2(i)(|x|^{2+2\beta}+|x|^2), \\ &\mathrm{tr}(\sigma_2(x,i)\sigma_2'(x,i)xx') \ge K_3(i)|x|^4, \\ &\mathrm{tr}(\sigma_2(x,i)\sigma_2'(x,i)) \le K_4(i)|x|^2. \end{aligned} \tag{19}$$

(d) The Markov chain $\alpha(t)$ is irreducible in the sense that the system of equations

$$\begin{cases} \nu Q = 0 \\ \nu \mathbb{1} = 1 \end{cases}$$

has a unique positive solution, where $\mathbb{1}$ is a column vector with all component being 1.

Example

Begin with (16) together with initial condition $X(0) = 1$. Suppose that $\alpha(t)$ is a Markov chain with two states $\mathscr{M} = \{1, 2\}$ and $Q = \begin{pmatrix} -0.1 & 0.1 \\ 1 & -1 \end{pmatrix}$, $f(x, 1) = x(x+1)$ and $f(x, 2) = x(2x+1)$.
Corresponding to the states, we have two equations

$$\begin{aligned} \frac{d}{dt}X(t) &= X(t)(X(t)+1), \\ \frac{d}{dt}X(t) &= X(t)(2X(t)+1). \end{aligned} \tag{20}$$

Neither equation has a global soln. For the 1st equation, we have $X(t) = e^t/(2 - e^t)$ that will blow up at time $\ln 2$; for the second equation, $X(t) = e^t/(3 - 2e^t)$ that will blow up at time $\ln(3/2)$. We plot the trajectories of the switched system.

To regularize the system, use a feedback control $a_1(\alpha(t))X^2(t)dw_1(t)$, where $w_1(t)$ is a 1-d Brownian motion. The resulting eq is

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))X^2(t)dw_1(t), \tag{21}$$

$a_1(i) = 2$ for $i = 1, 2$.
Although the system has a global solution, it is not asymptotically stable. To stabilize the system, we add another feedback control $a_2(\alpha(t))X(t)dw_2(t)$, $w_2(t)$ is 1-d standard Brownian motion independent of $w_1(t)$ and $a_2(1) = 19$ and $a_2(2) = 24$.

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))X^2(t)dw_1(t) + a_2(\alpha(t))X(t)dw_2(t). \tag{22}$$

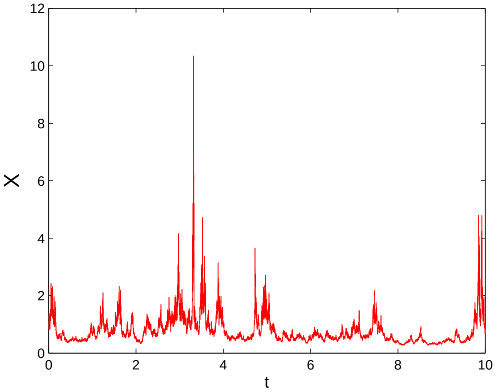


Figure: Trajectory of system (21) with stepsize $\Delta t = 10^{-4}$.

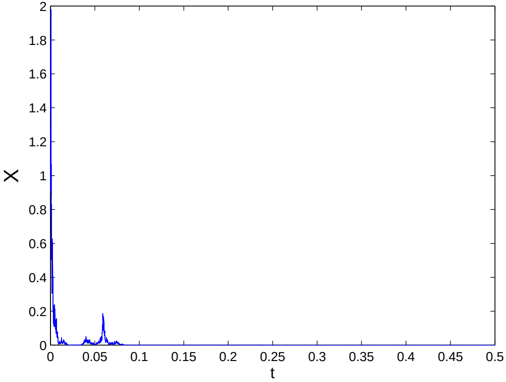
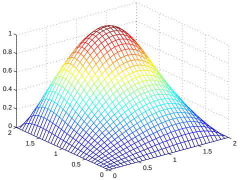
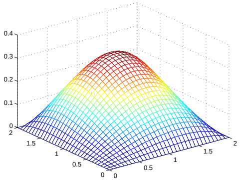


Figure: Trajectory of system (22) with stepsize $\Delta t = 10^{-6}$.

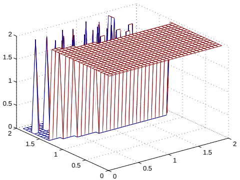
Numerical Methods for Control and Games



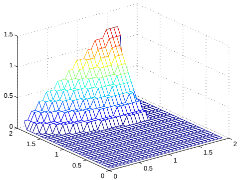
(a) $V^{h,+}(\cdot,\cdot,1)$



(b) $V^{h,+}(\cdot,\cdot,2)$



(c) $U_1(\cdot,\cdot,1)$: player1 1st



(d) $U_1(\cdot,\cdot,2)$ player1 1st

Numerics for Controlled Switching Diffusions

$$\begin{cases} X(t) = x + \int_0^t b(X(s), \alpha(s), u(s))ds + \int_0^t \sigma(X(s), \alpha(s))dw, \\ \alpha(t) \;\; \text{continuous-time MC} \;\; \alpha(0) = i, \end{cases} \quad (23)$$

where $w(t)$ is a standard Brownian motion independent of the Markov chain $\alpha(t)$.

- Kushner & Dupuis, Springer, Markov chain approximation
- with Song & Zhang, (2006), regime-switching & jump diffusion

Controlled Switching Diffusions (cont.)

Given $B > 0$, define a stopping time as

$$\tau_B^{x,i,u} = \inf\{t : X^{x,i,u}(t) \notin (-B, B)\}.$$

Objective: choose control $u.$ to minimize the expected cost function

$$\begin{cases} J_i^B(x,u) & = \mathbf{E} \displaystyle\int_0^{\tau_B^{x,i,u}} f(X(s), \alpha(s), u(s))ds, \\ & \qquad \forall x \in (-B,B),\ i \in \mathscr{M}, \\ J_i^B(x,u) & = 0,\ \forall x \notin (-B,B),\ i \in \mathscr{M}, \end{cases} \tag{24}$$

where for each $i \in \mathscr{M}$, $f(\cdot, i, \cdot)$ is an appropriate function representing the running cost function.

For each $i \in \mathscr{M}$, the value function is given by

$$V^B(x,i) = \inf_{u \in \mathscr{U}} J^B(x,i,u), \tag{25}$$

where $\mathscr{U}$ is the space of all $\mathscr{F}_t$-adapted controls taking values on a compact set U.

Formally, the value functions satisfy Hamilton-Jacobi-Bellman (HJB) equations,

$$\begin{cases} \inf_{u \in U}\{L^u V^B(x,i) + f(x,i,u)\} = 0, & \forall x \in (-B,B), i \in \mathscr{M}, \\ V^B(x,i) = 0, & \forall x \notin (-B,B), i \in \mathscr{M}, \end{cases} \tag{26}$$

where

$$L^u \varphi(x,i) = \frac{1}{2}\sigma^2(x,i)\frac{d^2\varphi(x,i)}{dx^2} + b(x,i,u)\frac{d\varphi(x,i)}{dx} + \sum_{j \in \mathscr{M}} q_{ij}\varphi(x,j).$$

Algorithm

- $h > 0$: discretization parameter.
- $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \ldots\}$. Let $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathscr{M}$
- $p^h((x,i),(y,j)|u)$: transition probabilities from $(x,i) \in S_h \times \mathscr{M}$ to $(y,j) \in S_h \times \mathscr{M}$, for $u \in U$.

Then, $\bar{V}^{B,h}(x,i)$, the discretization of $V^B(x,i)$ with step size $h>0$, is the solution of

$$\begin{cases} \inf_{u\in U}\{L_h^u \bar{V}^{B,h}(x,i)+f(x,i,u)\}=0, & \forall x\in(-B,B)_h,\ i\in\mathscr{M}, \\ \bar{V}^{B,h}(x,i)=0, & \forall x\notin(-B,B)_h,\ i\in\mathscr{M}, \end{cases} \tag{27}$$

where

$$(-B,B)_h=(-B,B)\cap S_h,\ \ [-B,B]_h=(-B,B)_h\cup\{B,-B\}. \tag{28}$$

$$\begin{aligned} \bar{V}^{B,h}(x,i) \ &= \inf_{u\in U}\Big\{\bar{p}_i^{h,+}(x,u)\bar{V}^{B,h}(x+h,i)+\bar{p}_i^{h,-}(x,u)\bar{V}^{B,h}(x-h,i) \\ &\qquad +\sum_{j\neq i}\bar{p}_{ij}^{h}(x)\bar{V}^{B,h}(x,j)+f(x,i,u)\Delta\bar{t}_i^h(x)\Big\} \end{aligned} \tag{29}$$

Rates of Convergence

Theorem

Under suitable conditions, $\exists \gamma > 2$ *and* $\rho \in (0,1]$ *s.t. the Markov chain approximation algorithm converges at the rate* $(\gamma - 2) \wedge \rho \wedge \frac{1}{2}$*. That is,*

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \le K h^{\frac{1}{2} \wedge \rho \wedge (\gamma - 2)}, \quad \forall (i,x) \in \mathcal{M} \times G.$$

- Note that $\gamma > 2$ comes from Markov chain $\approx$ for switching, ρ is the Hölder exponent of the cost function.
- PDE approach for controlled diffusions (finite difference approx of PDEs)
  - Menaldi, SIAM J. Control Optim. (1989)
  - Krylov, Probab. Theory Related Fields, (2000)
  - Dong & N.V. Krylov, Appl. Math Optim.
- We use probabilistic approach for controlled switching diffusions

Main Ideas (work with Q.S. Song, probabilistic approach)

- Use relaxed controls (measures)
- Construct approximation sequence
- Consider boundary perturbations
  - usual notion of cost $J_i(x, \widetilde{m})$;
  - ours $J_i^{B}(x, \widetilde{m})$

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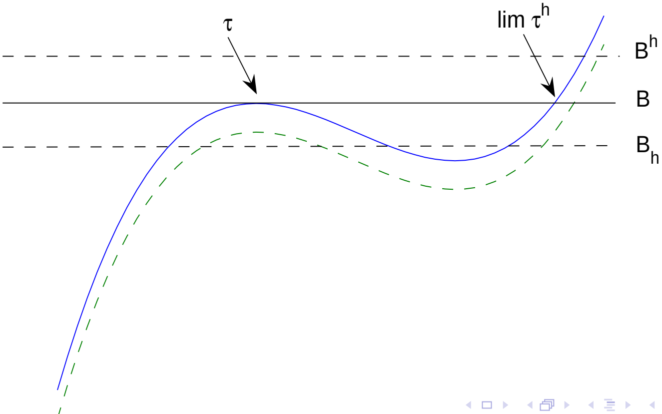
Tangency Problem

- τ and τ^h: the first hitting time of $X(t)$ and $x^h(t)$ to the boundary.
- Objective: $\approx \mathbf{E}\tau$ by $\mathbf{E}\tau^h$
- In the Figure, $\tau^h \not\to \tau$, even though $x^h(\cdot)$ converges to $X(\cdot)$.
- Q: extra conditions needed?



Thank you