Path Independent Property of the Action Functionals for Stochastic Dynamical Systems

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Outline

Given $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,\infty)})$. Consider a stochastic dynamical system described by the following SDE of the Markovian type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

where
$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$,
$\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, and
B_t is d-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion.
It is well known that under the usual conditions of linear growth and locally Lipschitz for the coefficients b and σ, there exists a unique solution to the equation with given initial data X_0.

The celebrated Girsanov theorem provides a very powerful tool to solve SDEs under the name of the *Girsanov transformation or the transformation of the drift*. Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the following condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right)\right] < \infty, \quad \forall t > 0.$$

Then, by Girsanov theorem,

$$\exp\left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right), \quad t \in [0, \infty)$$

is an $\{\mathcal{F}_t\}$-martingale. Furthermore, for $t \geq 0$, we define

$$Q_t := \exp\left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right) \cdot P$$

or equivalently in terms of the Radon-Nikodym derivative

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right).$$

Then, for any $T > 0$,

$$\tilde{B}_t := B_t - \int_0^t \gamma(s, X_s)ds, \quad 0 \leq t \leq T$$

is an $\{\mathcal{F}_t\}$-Brownian motion under the probability Q_T.
Moreover, X_t satisfies

$$dX_t = [b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)]dt + \sigma(t, X_t)d\tilde{B}_t, \quad t \geq 0.$$

Motivation from economics and finance Now look at

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \right)$$

we see that generally $\frac{dQ_t}{dP}$ depends on the "history" of the path up to t (i.e., $\{X_s : 0 \leq s \leq t\}$)! While in economics and finance studies, in particular towards to the optimal problem for the utility functions in an equilibrium market, it is a necessary requirement that $\frac{dQ_t}{dP}$ depends only on the state X_t, not on the whole "history" $\{X_s : 0 \leq s \leq t\}$. See, e.g., [1] E. Stein, J.C. Stein: Stock price distributions with stochastic volatility: an analytic approach. *The Review of Financial Studies* **4** (1991), 727-752; [2] S. Hodges, A. Carverhill: Quasi mean reversion in an efficient stock market: the characterisation of Economic equilibria which support Black-Scholes Option pricing. *The Economic Journal* **103** (1993), 395-405.

So mathematically, one requires that the Radon-Nikodym derivative is in the form of

$$Z(X_t, t) = \frac{dQ_t}{dP}, \quad t \in [0, \infty).$$

We call this the *path-independent property* of the density of the Girsanov transformation. A characterisation of this property for the above SDEs was obtained in

A. Truman, F.-Y. Wang, J.-L. Wu and W. Yang: A link of stochastic differential equations to nonlinear parabolic equations, *SCIENCE CHINA Mathematics*, in press.

Assumptions:
(i) (*Non-degeneracy*) The coefficient σ satisfies that the matrix $\sigma(t,x)$ is invertible, for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$;

(ii) Specify the function γ by

$$\gamma(t,x) = -(\sigma(t,x))^{-1} b(t,x)$$

so that $b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t) = 0$, and hence we require b and σ satisfy

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |(\sigma(s,X_s))^{-1} b(s,X_s)|^2 ds\right)\right] < \infty, \quad \forall t > 0.$$

Thus the associated probability measure Q_t is determined by

$$\frac{dQ_t}{dP} = \exp\left(-\int_0^t \langle(\sigma(s,X_s))^{-1}b(s,X_s), dB_s\rangle - \frac{1}{2}\int_0^t \left|(\sigma(s,X_s))^{-1}b(s,X_s)\right|^2 ds\right).$$

Now set

$$\hat{Z}_t := -\ln\frac{dQ_t}{dP}$$

that is

$$\hat{Z}_t = \int_0^t \langle(\sigma(s,X_s))^{-1}b(s,X_s), dB_s\rangle + \frac{1}{2}\int_0^t \left|(\sigma(s,X_s))^{-1}b(s,X_s)\right|^2 ds.$$

Clearly, $\hat{Z}_t$ is a one dimensional stochastic process with the stochastic differential form

$$d\hat{Z}_t = \frac{1}{2}\left|(\sigma(t,X_t))^{-1}b(t,X_t)\right|^2 dt + \langle(\sigma(t,X_t))^{-1}b(t,X_t), dB_t\rangle.$$

Theorem (Characterisation Theorem)

Let $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \frac{1}{2} \int_0^t \left|(\sigma(s, X_s))^{-1} b(s, X_s)\right|^2 ds \\ &\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned}$$

equivalently,

$$\frac{dQ_t}{dP} = \exp\{v(0, X_0) - v(t, X_t)\}, \quad t \in [0, \infty)$$

holds if and only if

Theorem (cont'd)

$$b(t,x) = (\sigma\sigma^*\nabla v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

and v satisfies the following time-reversed Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\left\{\left[Tr(\sigma\sigma^*\nabla^2 v)\right](t,x) + |\sigma^*\nabla v|^2(t,x)\right\}$$

where $\nabla^2 v$ stands for the Hessian matrix of v with respect to the second variable.

Proof

Necessity Assume that there exists a scalar function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ which is C^1 with respect to the first variable and C^2 with respect to the second variable such that

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \frac{1}{2}\int_0^t \left|(\sigma(s, X_s))^{-1} b(s, X_s)\right|^2 ds \\ &\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned}$$

holds, then we have

$$dv(t, X_t) = \frac{1}{2}\left|(\sigma(t, X_t))^{-1} b(t, X_t)\right|^2 dt + \langle (\sigma(t, X_t))^{-1} b(t, X_t), dB_t \rangle .$$

Now viewing $v(t, X_t)$ as the composition of the deterministic $C^{1,2}$-function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ with the continuous semi-martingale X_t, we can apply Itô's formula to $v(t, X_t)$ and further with the help of our original SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

we have the following derivation

$$\begin{aligned} dv(t, X_t) &= \left\{ \frac{\partial}{\partial t} v(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, X_t) \right. \\ &\quad \left. + \langle b, \nabla v\rangle(t, X_t) \right\} dt + \langle(\sigma^* \nabla v)(t, X_t), dB_t\rangle \end{aligned}$$

since

$$\langle \nabla v(t, X_t), \sigma(t, X_t)dB_t\rangle = \langle \sigma^*(t, X_t)\nabla v(t, X_t), dB_t\rangle .$$

Now comparing this with the previously obtained

$$dv(t,X_t) = \frac{1}{2}\left|(\sigma(t,X_t))^{-1}b(t,X_t)\right|^2 dt + \langle(\sigma(t,X_t))^{-1}b(t,X_t), dB_t\rangle$$

and using the uniqueness of Doob-Meyer's decomposition of continuous semi-martingale, we conclude that the coefficients of dt and dB_t must coincide, respectively, namely

$$(\sigma^{-1}b)(t,X_t) = (\sigma^*\nabla v)(t,X_t)$$

$$\frac{1}{2}|(\sigma^{-1}b)(t,X_t)|^2 = \frac{\partial}{\partial t}v(t,X_t) + \frac{1}{2}[Tr(\sigma\sigma^*\nabla^2 v)](t,X_t) + \langle b, \nabla v\rangle(t,X_t)$$

holds for all $t > 0$.

Since our SDE is non-degenerate, the support of $X_t, t \in [0, \infty)$ is the whole space $\mathbb{R}^d$. Hence, the following two equalities

$$(\sigma^{-1}b)(t,x) = (\sigma^* \nabla) v(t,x)$$

$$\frac{1}{2}|(\sigma^{-1}b)(t,x)|^2 = \frac{\partial}{\partial t} v(t,x) + \langle b, \nabla v \rangle (t,x) + \frac{1}{2}[Tr(\sigma\sigma^* \nabla v)](t,x)$$

hold on $[0, \infty) \times \mathbb{R}^d$. From these equalities we derive

$$b(t,x) = (\sigma\sigma^* \nabla v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

and v satisfies the Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\left\{ \left[Tr(\sigma\sigma^* \nabla^2 v) \right](t,x) + |\sigma^* \nabla v|^2 (t,x) \right\} .$$

Sufficiency Assume that there exists a $C^{1,2}$ scalar function $v : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ solving the Burgers-KPZ equation. Specify the drift b of the original SDE via

$$b(t,x) = (\sigma\sigma^* \nabla v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d .$$

We then have

$$\begin{aligned} dv(t,X_t) &= \Big[-\frac{1}{2}|\sigma^* \nabla v|^2(t,X_t) + \langle b, \nabla v\rangle (t,X_t)\Big] dt \\ &\qquad + \langle (\sigma^* \nabla v)(t,X_t), dB_t\rangle \\ &= \frac{1}{2}|\sigma^{-1} b|^2 (t,X_t) dt + \langle (\sigma^{-1} b)(t,X_t), dB_t\rangle . \end{aligned}$$

This clearly implies

$$
\begin{aligned}
v(t, X_t) \quad = \quad & v(0, X_0) + \frac{1}{2} \int_0^t \left| (\sigma(s, X_s))^{-1} b(s, X_s) \right|^2 ds \\
& + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle
\end{aligned}
$$

by taking stochastic integration. This completes the proof.

Path-independent phenomenon also appeared in Calculus of Variation and Stochastic Optimal Control. Here some ref's

[1] J.C. Zambrini: Variational processes and stochastic versions of mechanics. *J Math Physics* **27** (1986), 2307-2330.
[2] A.B. Cruzeiro, J.C. Zambrini: Malliavin calculus and Euclidean quantum mechanics. *J Funct Anal* **96** (1991), 62-95.
[3] A.B. Cruzeiro, L. Wu, J.C. Zambrini: Bernstein processes associated with a Markov process. pp41-72 in *Stochastic Analysis and Mathematical Physics*, Birkhäuser, 2000.
[4] K.L.Chung, J.C. Zambrini: *Introduction to Random Time and Quantum Randomness*. World Scientific, 2003.
[5] W.H. Fleming, H.M. Soner: *Controlled Markov Processes and Viscosity Solutions*. (2nd Ed) Springer, 2006.
[6] J.C. Zambrini: On the geometry of the Hamilton-Jacob-Bellman equation. *J Geometric Mechanics* **1** (2009), 369-387.
[7] N. Privault, J.C. Zambrini: Stochastic deformation of integrable dynamical systems and random time symmetry. *J Math Phys* **51** (2010), 082104.

Given a probability space $(\Omega, \mathcal{F}, P)$. We are concerned with a stochastically deformed dynamical system described as the following n-dimensional Itô stochastic differential equation

$$\begin{cases} dZ(\tau) = B(Z(\tau), \tau)d\tau + h^{\frac{1}{2}} dW(\tau), \quad \tau \in (t, u] \\ Z(t) = q \in \mathbb{R}^n \end{cases} \tag{1}$$

with $0 \leq t < u < \infty$ and $q \in \mathbb{R}^n$ being arbitrarily fixed, where $B : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is a vector field, $h > 0$ is the deformation parameter and $W = \{W(\tau)\}_{\tau \geq 0}$ is an n-dimensional standard Brownian motion . It is well known that under the usual conditions of linear growth and locally Lipschitz in the space variable for the drift coefficient B, there exists a unique solution to the equation together with the given initial data $Z(t) = q$.

Actually, the solution $Z = \{Z(\tau)\}_{\tau \in [t,u]}$ of the initial value problem (1) is a $\mathbb{R}^n$-valued diffusion process on $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\{\mathcal{F}_\tau\}_{\tau \in [0,\infty)}$ generated by the Brownian motion $\{W(\tau)\}_{\tau \in [t,u]}$ in the sense that $\mathcal{F}_r$ represents the past information generated by Z up to any time $r \in [t, u]$. If, moreover, we specify Ω as the path space of Z to be the classical Wiener space (endowed with the supremum norm)

$$\Omega = \mathcal{W}([t,u]) := \{\omega \in C([t,u] \to \mathbb{R}^n) : \omega(t) = q \,\text{and}\, \omega(u) = y \,\text{fixed}\} \tag{2}$$

then the process Z can be realised canonically as

$$Z(\tau, \omega) = \omega(\tau), \quad \tau \in [t, u], \ \omega \in \Omega .$$

Such a stochastic deformation of dynamical system belongs to the class of Bernstein or reciprocal processes initiated in Zambrini [1] and developed further in [2,3,4].

Furthermore, for any such diffusion process $Z(\tau), \tau \in [t, u]$, we can define its Kolmogorov (infinitesimal) generator in the following manner

$$\begin{aligned} Df(Z(\tau),\tau) &:= \lim_{\theta \to 0} \mathbb{E}_{q,t}\left[\frac{f(Z(\tau+\theta),\tau+\theta) - f(Z(\tau),\tau)}{\theta}\right] \\ &= \frac{\partial f}{\partial \tau}(Z(\tau),\tau) + B(Z(\tau),\tau)\nabla f(Z(\tau),\tau) \\ &\quad + \frac{h}{2}\Delta f(Z(\tau),\tau) \end{aligned} \tag{3}$$

for any function $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ in the domain such that the above limit exists. In particular, taking for f any component q_j of $q = (q_1, q_2, ..., q_n) \in \mathbb{R}^n$, one derives

$$DZ(\tau) = B(Z(\tau), \tau)$$

Due to the well known fact that, with probability one, the sample paths of Brownian motion W is nowhere differentiable (cf. e.g. the seminal paper in 1961 by Dvoretzky, Erdös and Kakutani), the diffusion process Z does not differentiate either. Thus, the classical action functional along the diffusion process Z becomes divergent so one can not consider the classical action functional for Z. However, with the help of the Kolmogorov generator D, the following action functional for the process $Z = \{Z_\tau\}_{\tau\in[t,u]}$ was introduced in Cruzeiro and Zambrini [2] (utilising Malliavin calculus)

$$F[Z] := \mathbb{E}_{q,t} \int_t^u L(Z(\tau), DZ(\tau))d\tau \tag{4}$$

with the classical Lagrangian

$$L(\omega, \dot\omega) = \frac{1}{2}|\dot\omega|^2 + V(\omega), \quad \omega \in C^1([0,\infty) \to \mathbb{R}^n). \tag{5}$$

where $\mathbb{E}_{q,t}$ denotes the conditional expectation given $Z(t) = q$, $\dot{\omega} := \frac{d\omega(\tau)}{d\tau}$ the time derivative of the path $\omega \in C^1([0,\infty) \to \mathbb{R}^n)$, and the potential $V : \mathbb{R}^n \to \mathbb{R}$ is a function in the Kato class. For F, its domain is defined in the following manner

$$\mathrm{Dom}(F) := \left\{ Z = (Z(\tau))_{\tau\in[t,u]} : \mathbb{E}_{q,t} \int_t^u |L(Z(\tau), DZ(\tau))| d\tau < \infty \right\}.$$

Recall that for the classical Wiener space $\mathcal{W}$ defined in (2), we have the Cameron-Martin space $\mathcal{H} \subset \mathcal{W}$ endowed with the scalar product

$$< \omega_1, \omega_2 >_{\mathcal{H}} := \int_t^u \dot{\omega}_1(\tau) \cdot \dot{\omega}_2(\tau) d\tau, \quad \omega_1, \omega_2 \in \mathcal{H}.$$

It is known that for any $\delta Z \in \mathcal{H}$ and for any $\epsilon > 0$, the (path) probability measure induced by the shifted process $(Z(\tau) + \epsilon\delta Z(\tau))_{\tau\in[t,u]}$ is absolutely continuous with respect to the path probability measure of $(Z(\tau))_{\tau\in[t,u]}$. Moreover, the variation $F[Z + \epsilon\delta Z] - F[Z]$ of $F[Z]$ is well defined, so we can define directional derivative (cf. [2,6,7] where the directional derivative can be even computed explicitly)

$$\nabla F[Z](\delta Z) := \lim_{\epsilon\to 0} \frac{F[Z + \epsilon\delta Z] - F[Z]}{\epsilon}$$

With all these in hand, one can then have the following

Definition

A process $Z \in \mathrm{Dom}(F)$ is called *a critical point* (or a minimal point) for F if the directional derivative of F at Z along any direction δZ vanishes in the following sense

$$\mathbb{E}_{q,t}\left(\nabla F[Z](\delta Z)\right) = 0\,.$$

It has been shown in [2] that any critical point Z of the action functional $F[Z]$ from the class of diffusion processes determined by (1) satisfies the deformed Euler-Lagrange equation (along the paths of Z)

$$D\left(\frac{\partial L(Z(\tau), DZ(\tau))}{\partial DZ(\tau)}\right) - \frac{\partial L(Z(\tau), DZ(\tau))}{\partial Z(\tau)} = 0$$

which is equivalent to

$$DDZ(\tau) = \nabla V(Z(\tau))$$

since

$$\frac{\partial L(Z(\tau), DZ(\tau))}{\partial Z(\tau)} = \frac{\partial}{\partial Z(\tau)} \left(\frac{1}{2}|DZ(\tau)|^2 + V(Z(\tau)) \right) = \nabla V(Z(\tau))$$

and

$$\frac{\partial L(Z(\tau), DZ(\tau))}{\partial DZ(\tau)} = \frac{\partial}{\partial DZ(\tau)} \left(\frac{1}{2}|DZ(\tau)|^2 + V(Z(\tau)) \right) = DZ(\tau)\,.$$

The above action functional $F[Z]$ also enjoys the following "invariance property". Let $S : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a $C^{2,1}$ scalar function. Along the paths of Z, one can define

$$DS(Z(\tau,\tau)) := \frac{\partial S(Z(\tau),\tau)}{\partial \tau} + DZ(\tau)\nabla S(Z(\tau),\tau) + \frac{h}{2}\Delta S(Z(\tau),\tau)$$

and build another action functional

$$\hat{F}[Z] := \mathbb{E}_{q,t} \int_t^u \hat{L}(Z(\tau), DZ(\tau))d\tau$$

with the associated Lagrangian

$$\hat{L}(Z(\tau), DZ(\tau)) := L(Z(\tau), DZ(\tau)) + DS(Z(\tau), \tau) \, .$$

Then it is shown in [1] that the action functional $\hat{F}$ is equivalent to the original action functional $F[Z]$ in the sense that their associated (deformed) Euler-Lagrange equations are the same. So DS has been given a name as *a deformed null Lagrangian*. In addition, if the scalar function S is a solution to the following Hamilton-Jacobi-Bellman equation

$$\frac{\partial S(z,\tau)}{\partial \tau} - \frac{1}{2}|\nabla S(z,\tau)|^2 + \frac{h}{2}\Delta S(z,\tau) + V(z) = 0$$

then along a critical point $Z = (Z(\tau))_{\tau \in [t,u]}$, we have $\hat{L}(Z(\tau), DZ(\tau)) = 0$ so

$$L(Z(\tau), DZ(\tau)) = \hat{L}(Z(\tau), DZ(\tau)) - DS(Z(\tau), \tau) = -DS(Z(\tau), \tau)$$

from which one can obtain Dynkin's formula (cf. e.g. K. Itô: *Stochastic Processes*, Springer, 2004.)

$$F[Z] = \mathbb{E}_{q,t} \int_t^u (-DS(Z(\tau), \tau)) d\tau = S(q, t) - \mathbb{E}_{q,t} S(Z(u), u).$$

showing that the action functional $F[Z]$ has *the path-independent property* with respect to Z. This is expected because the critical point is defined as the one minimises the action functional $F[Z]$ over all $Z's$.

A.B. Cruzeiro, J.-L. Wu and J.-C. Zambrini: Stochastically integrable systems and stochastic Stokes formula, working paper.

Our objective here is to characterise the path-independence of the action functionals.

Definition

We say that the action functional $F[Z]$ possesses the path-independent property if there is a scalar $C^{2,1}$-function $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ such that

$$F[Z] := \mathbb{E}_{q,t} \int_t^u L(Z(\tau), DZ(\tau))d\tau = f(q, t) - \mathbb{E}_{q,t} f(Z(u), u).$$

Theorem

(i) Let $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be $C^{2,1}$. Then

$$F[Z] := \mathbb{E}_{q,t} \int_t^u L(Z(\tau), DZ(\tau))d\tau = f(q,t) - \mathbb{E}_{q,t} f(Z(u), u)$$

if and only if the scalar function $S : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ defined by

$$S(z,\tau) := \frac{1}{2}|B(z,\tau)|^2 + V(z) - f(z,\tau)$$

satisfies the following Hamilton-Jacobi-Bellman equation

$$\frac{\partial S(z,\tau)}{\partial \tau} - \frac{1}{2}|\nabla S(z,\tau)|^2 + \frac{h}{2}\Delta S(z,\tau) + V(z) = 0.$$

Theorem (cont'd)

(ii) If, moreover, Z is a critical point for F, then the path-independent property of the action functional $F[Z]$ is characterised by its drift vector field B given in the form

$$B(z,\tau) = h\nabla \log \eta(z,\tau), \quad (z,\tau) \in \mathbb{R}^n \times [0,\infty)$$

where $\eta : (z,\tau) \in \mathbb{R}^n \times [0,\infty) \mapsto \eta(z,\tau) \in (0,\infty)$ satisfies the following linear parabolic equation

$$h\frac{\partial \eta(z,\tau)}{\partial \tau} = -\frac{h^2}{2}\Delta\eta(z,\tau) + V(z)\eta(z,\tau), \quad \tau \in [t,u].$$

Thank You!