The hitting problems for the reflected O-U processes

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Outline

Motivations: In this talk, we re-visit and discuss the hitting problems on a class of reflected Ornstein-Uhlenbeck (abbr. O-U) processes, due to their growing applications in Queueing Systems, Financial Modelings and some other fields. The reflected Ornstein-Uhlenbeck processes are typical Markov processes, and especially the mean-reverting stationary process. With some dedicate performances, we are able to eventually get the explicit and practical results for those Laplace transforms of the hitting probabilities.

- First there have been a wide applications of reflected O-U processes in the modeling of Queueing Systems and Supply Chains (see, e.g. Harrison (1986), $\cdots$).
- Now there is also a growing applications in the Financial Modeling, for instance, for modeling the Exchange rate target zone (see e.g. Svensson (1991;1992), Larsen and Sorensen (2007), and even Krugman (1991)), and for modeling the Interest rate term-structure (see, e.g. Goldstein and Keirstead (1997)) and the others in the literature. The reflected processes are especially incorporate for modeling the regulated market price dynamics.

- The another advantage of using the processes for the modelings is that, comparative to some other stochastic processes, the reflected Ornstein-Uhlenbeck process exhibits a good property for the tractability due to the few parameters could determining the long term process paths. Actually we had some performance in the parameters estimates on the processes (see below).

Maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes (Bo-Yang-Wang-Zhang). *J. Stat. Plann. Inference* **141**:1 (2011).

- Let's recall the Ornstein-Uhlenbeck (O-U) process, which is the solution of such a SDE:

$$\mathrm{d}X_t = (\mu - \alpha X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad X_0 \in \mathbf{R},$$

where $\mu \in \mathbf{R}$, $\alpha > 0$ and $\sigma > 0$.

- and the two-sided reflected (or regulated) Ornstein-Uhlenbeck process, which is the solution of such a reflected SDE:

$$\mathrm{d}X_t = (\mu - \alpha X_t)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t - \mathrm{d}U_t, \quad X_0 \in [a, b],$$

Here a and b are two barriers. L and U are the respective forces for reflecting the process at barriers a and b, and for keeping the process path within the interval $[a, b]$.

- Now we start at a reflected O-U process X. Let's define the First Passage Time of the process X across a barrier l by

$$T_l = \inf\{t > 0;\ X_t = l\}, \qquad l \in [0, b].$$

- The central aim is to compute the Laplace transform (LT) of T_l:

$$f_\theta(x; l) = \mathbf{E}_x \left[\exp\left(-\theta T_l\right) \mathbf{1}_{\{T_l < \infty\}}\right],$$

where the initial point $x \in [0, b]$ and for $\theta > 0$. This will be done in the sections below.

We consider the two-sided reflected O-U process and we are going to compute the LT of the First Passage time. For this purpose, we consider the equations:

$$\frac{1}{2}\sigma^2 f_1''(x) + (\mu - \alpha x) f_1'(x) - \theta f_1(x) = 0$$

with smooth pasting condition $f_1'(0) = 0$ at lower barrier,

and

$$\frac{1}{2}\sigma^2 f_2''(x) + (\mu - \alpha x) f_2'(x) - \theta f_2(x) = 0$$

with smooth pasting condition $f_2'(b) = 0$ at upper barrier.

Then we can get

Theorem (BWZ , Queueing Syst. 2006)

The LT function $f_\theta(x;l)$ is given by

$$f_\theta(x;l) = \frac{f_1(x)}{f_1(l)}, \quad \text{if } x \leq l \leq b,$$

$$f_\theta(x;l) = \frac{f_2(x)}{f_2(l)}, \quad \text{if } 0 \leq l \leq x.$$

Further a numerical approximation argument could eventually give $f_\theta(x;0)$ and $f_\theta(x;b)$. In the following, we give an example for illustration.

Example

Assume that $\mu = 0$, $\sigma = 1$, and $0 \le l \le x \le b$. Let $D_{-q}(y)$ be a Parabolic Cylinder Function, defined by:

$$D_{-q}(y) = \frac{e^{-x^2/4}}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-t^2/2 - yt} \mathrm{d}t, \quad y \in \mathbf{R},\ q > 0.$$

Then there is an explicit expression,

$$f_\theta(x;l) = \exp\left(\frac{\alpha}{2\sigma^2}(x^2 - l^2)\right)$$

$$\times \quad \frac{D_{-\theta/\alpha}\left(-\frac{x\sqrt{2\alpha}}{\sigma}\right) D_{-1-\theta/\alpha}\left(\frac{b\sqrt{2\alpha}}{\sigma}\right) + D_{-\theta/\alpha}\left(\frac{x\sqrt{2\alpha}}{\sigma}\right) D_{-1-\theta/\alpha}}{D_{-\theta/\alpha}\left(-\frac{l\sqrt{2\alpha}}{\sigma}\right) D_{-1-\theta/\alpha}\left(\frac{b\sqrt{2\alpha}}{\sigma}\right) + D_{-\theta/\alpha}\left(\frac{l\sqrt{2\alpha}}{\sigma}\right) D_{-1-\theta/\alpha}}$$

Reflected Markov-Modulated SDE

- We next switch to consider a reflected Markov-Modulated Process with two-sided barriers a and b, which is formulated by:

$$dX_t = \mu(\varepsilon_t, X_t)dt + \sigma(\varepsilon_t, X_t)dW_t + dL_t - dU_t, \quad X_0 \in [a, b],$$

where ε is a continuous-time Markov Chain with finite states, but which is independent of W.

Reflected Markov-Modulated SDE

- Analogically, define the first passage time T_l, and then formulate its LT by

$$f_{ij}^{\theta}(x;l) := \mathbf{E}_{x,i}\left[\exp\left(-\theta T_l\right)\mathbf{1}_{\{\varepsilon_{T_l}=j,T_l<\infty\}}\right],$$

  where $x, l \in [a, b]$, and $i, j \in S$, the state space of the Markov chain.

- In particular, set $S = \{0, 1\}$ and $\lambda_0, \lambda_1 > 0$, and define

$$\begin{aligned}\mathcal{A}_i f(x;l) \quad =: \quad & \frac{1}{2}\sigma^2(i,x)f''(x;l) - \mu(i,x)f_x(x;l) \\ & +(\theta+\lambda_i)f(x;l).\end{aligned}$$

  Then

Theorem (BWY , 2012)

For $a \le x < l$, the LT functions $f_{ij}^{\theta}(x;l)$ are determined by

$$\mathcal{A}_0 f_{00}^{\theta}(x;l) = \lambda_0 f_{10}^{\theta}(x;l), \qquad \mathcal{A}_1 f_{10}^{\theta}(x;l) = \lambda_0 f_{00}^{\theta}(x;l),$$
$$\mathcal{A}_1 f_{11}^{\theta}(x;l) = \lambda_1 f_{01}^{\theta}(x;l), \qquad \mathcal{A}_0 f_{01}^{\theta}(x;l) = \lambda_0 f_{11}^{\theta}(x;l).$$

with the boundary conditions

$$f_{10}^{\theta}(l;l) = f_{01}^{\theta}(l;l) = 1 - f_{00}^{\theta}(l;l) = 1 - f_{11}^{\theta}(l;l) = 0$$
$$f_{ij,x}^{\theta}(a;l) = 0, \ \text{ for } \ i,j = 0,1.$$

Remark: *For $l \le x \le b$, the conclusion of theorem also holds, if replacing $f_{ij,x}^{\theta}(b;l) = 0$ by $f_{ij,x}^{\theta}(a;l) = 0$ in the conditions.*

O-U Processes over Random Boundary

- We now go back to the Ornstein-Unlenbeck process:

$$\mathrm{d}X_t = (\mu - \alpha X_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where $\mu \in \mathbf{R}$ and $\alpha > 0$.

- Consider a random jump boundary $C_1 = (C_1(t);\ t \geq 0)$, which is given by

$$C_1(t) = b + Y_1 \mathbf{1}_{\{T_1 \leq t\}},$$

where $b > 0$. The r.v. T_1 admits an exponential law with the parameter $q > 0$, independent of W. Moreover assume that the r.v. Y_1 is independent of (T, W), with a distribution $F(\mathrm{d}y)$.

- The first passage time over random boundary is defined by

$$T_{0,b} = \inf\{t \geq 0;\ X_t = 0 \text{ or } X_t = C_1(t)\}.$$

- To calculate the joint LT of $(X_{T_{0,b}}, T_{0,b})$. Define the joint LT function as

$$f(\zeta, \theta; x) := \mathbf{E}_x\left[\exp\left(-\zeta X_{T_{0,b}} - \theta T_{0,b}\right)\right],$$

where $x \in (0, b)$ and $\zeta, \theta > 0$.

- Now we need to introduce the so-called Improper LTs of the First Passage Time for the OU process:

$$\begin{aligned} \varphi_1(\theta; b, x) &:= \mathbf{E}_x \left[\exp\left(-\theta\sigma_0\right) \mathbf{1}_{\{\sigma_0 < \sigma_b\}} \right], \\ \varphi_2(\theta; b, x) &:= \mathbf{E}_x \left[\exp\left(-\theta\sigma_b\right) \mathbf{1}_{\{\sigma_0 > \sigma_b\}} \right], \end{aligned}$$

where $\sigma_c = \inf\{t \geq 0;\ X_t = c\}$.

Lemma

The Improper LTs can be given by

$$\varphi_1(\theta; b, x) = \frac{f_\theta(x)\gamma(\theta; 0, b) - f_\theta(b)\gamma(\theta; 0, x)}{f_\theta(0)\gamma(\theta; 0, b) - f_\theta(b)},$$

and

$$\varphi_2(\theta; b, x) = \frac{f_\theta(0)\gamma(\theta; 0, x) - f_\theta(x)}{f_\theta(0)\gamma(\theta; 0, b) - f_\theta(b)},$$

where, $0 < x < b$,

$$f_\theta(x) = H_{-\frac{\theta}{\alpha}}\left(-\frac{\sqrt{\alpha}}{\sigma}(x - \frac{\mu}{\alpha})\right),$$

Lemma (Cont.)

and

$$\gamma(\theta; c, x) = \begin{cases} \dfrac{H_{-\frac{\theta}{\alpha}}\left(-\frac{\sqrt{\alpha}}{\sigma}(x - \frac{\mu}{\alpha})\right)}{H_{-\frac{\theta}{\alpha}}\left(-\frac{\sqrt{\alpha}}{\sigma}(c - \frac{\mu}{\alpha})\right)}, & \text{for } x < c, \\ \dfrac{H_{-\frac{\theta}{\alpha}}\left(\frac{\sqrt{\alpha}}{\sigma}(x - \frac{\mu}{\alpha})\right)}{H_{-\frac{\theta}{\alpha}}\left(\frac{\sqrt{\alpha}}{\sigma}(c - \frac{\mu}{\alpha})\right)}, & \text{for } x > c. \end{cases}$$

The function $H_\nu(x)$ *denotes the Hermite Function.*

- We also consider the LTs evaluated at exponential time T_1, which is expressed as follows:

$$\begin{aligned}\phi_0(\alpha,\theta;x) &:= \mathbf{E}_x\left[\exp\left(-\zeta X_{T_1}-\theta T_1\right)\right],\\ \phi_1(\alpha,\theta;x) &:= \mathbf{E}_x\left[\exp\left(-\zeta X_{T_1}-\theta T_1\right)\mathbf{1}_{\{T_{0,b}<T_1\}}\right].\end{aligned}$$

Theorem (BWY, JAP 2011)

For $x \in \mathbf{R}$*, and* $\zeta > 0$*, it holds that*

$$\phi_0(\zeta,\theta;x) = \frac{q}{\zeta}\int_0^{\zeta} \frac{e^{w(\zeta)-xz-w(z)}}{z}\mathrm{d}z,$$

where $w(\zeta) = -\frac{\zeta\mu}{\alpha} + \frac{\zeta^2\sigma^2}{4r} - \frac{q+\theta}{\alpha}\ln\zeta$*. And , for* $0 < x < b$*,*

$$\phi_1(\zeta,\theta;x) = \phi_0(\zeta,\theta;0)\varphi_1(q+\theta;b,x) + \phi_0(\zeta,\theta;b)\varphi_2(q+\theta;b,x),$$

where $\varphi_1(q+\theta;b,x)$ *and* $\varphi_2(q+\theta;b,x)$ *determined in the above Lemma.*

Theorem (BWY , JAP 2011)

For $0 < x < b$, *the LT function of* $(X_{T_{0,b}}, T_{0,b})$ *can be given by*

$$\begin{aligned}\psi(\zeta, \theta; x) &= \varphi_1(q+\theta; b, x) + e^{-\zeta b}\varphi_2(q+\theta; b, x) \\ &\quad + \mathbf{E}_x\left[e^{-\theta T_1} h(\theta; X_{T_1}, Y_1)\mathbf{1}_{\{T_{0,b} > T_1\}}\right],\end{aligned}$$

where

$$h(\theta; x, y) = \varphi_1(\theta; b+y, x) + e^{-\zeta(b+y)}\varphi_2(\theta; b+y, x)$$

- We now consider a one-side reflected O-U process in $[0, \infty)$ satisfying

$$dX_t = (\mu - \alpha X_t)dt + \sigma dW_t + dL_t, \quad X_0 \geq 0.$$

We can give alterative interpretation on the regulator.

Lemma

The regulator $L = (L_t;\ t \geq 0)$ *has the following expression.*

$$L_t = \lim_{\varepsilon \to 0} \frac{\sigma^2}{2\varepsilon} \int_0^t \mathbf{1}_{\{0 \leq X_s \leq \varepsilon\}} ds, \quad t \geq 0.$$

- **T**he transition density $p(t; x, y)$ of a one-side reflected O-U process X can be found in Linetsky (2005) (see the Section 5.2 therein). The LT of the First Passage Time $\hat{\sigma}_b = \inf\{t \geq 0;\ X_t = b\}$ is defined by

$$\hat{\gamma}(\theta; b, x) := \mathbf{E}_x \left[\exp\left(-\theta\hat{\sigma}_b\right)\right].$$

  We could get its explicit expression (see,e.g. BWZ (2006;2009)).
- **H**ereafter, we will regard $p(t; x, y)$ and $\hat{\gamma}(\theta; b, x)$ as known.

- We consider the First Passage Time of the one-side Reflected O-U process over the one-jump boundary C_1:

$$\hat{T}_b = \inf\{t \geq 0;\ X_t = C_1(t)\}.$$

- Similarly as in the case without reflection, we calculate LTs evaluated at exponential time T_1:

$$\begin{aligned}\hat{\phi}_0(\zeta,\theta;x) &:= \mathbf{E}_x\left[\exp\left(-\zeta X_{T_1} - \theta T_1\right)\right],\\ \hat{\phi}_1(\zeta,\theta;x) &:= \mathbf{E}_x\left[\exp\left(-\zeta X_{T_1} - \theta T_1\right)\mathbf{1}_{\{\hat{T}_b < T_1\}}\right].\end{aligned}$$

Theorem (BWY, JAP 2011)

For $0 < x < b$, *it holds that*

$$\hat{\phi}_0(\zeta,\theta;x) = \frac{q}{\alpha}\int_0^{\alpha}\frac{e^{w(\zeta)-xz-w(z)}}{z}\mathrm{d}z + \frac{\sigma^2}{2\alpha}\hat{g}_L(\theta;x)\int_0^{\zeta} e^{w(\zeta)-w(z)}\mathrm{d}z,$$

Theorem (Cont.)

where

$$\hat{g}_L(\theta; x) = \mathbf{E}_x \left[\exp\left(-\theta T_1\right) p(T_1; x, 0) \right],$$

and $p(t; x, 0)$ *is the transition density at* $y = 0$ *of the reflected O-U process. Moreover*

$$\hat{\phi}_1(\zeta, \theta; x) = \hat{\phi}_0(\zeta, \theta; b)\hat{\gamma}(q + \theta; b, x).$$

Theorem (BWY, JAP 2011)

For $0 < x < b$, the LT of $(X_{\hat{T}_b}, \hat{T}_b)$ is given by

$$\begin{aligned} \hat{\psi}(\zeta, \theta; x) &= e^{-\zeta b}\hat{\gamma}(q+\theta; b, x) \\ &\quad + e^{-\zeta b}\mathbf{E}_x\left[e^{-\zeta Y_1 - \theta T_1}\hat{\gamma}(\theta; b + Y_1, X_{T_1})\mathbf{1}_{\{\hat{T}_b > T_1\}}\right], \end{aligned}$$

where the second term in the r.h.s. of the above formula can be determined by the last theorem.

ROU Process with Jumps

- We now consider a reflected O-U process with jumps:

$$\mathrm{d}X_t = -\gamma X_t \mathrm{d}t + \mathrm{d}Z_t + \mathrm{d}L_t, \quad X_0 \geq 0,$$

  where $\gamma > 0$ and $Z = (Z_t;\ t \geq 0)$ is a spectrally negative $\alpha \in (1, 2]$-stable process with Laplace exponent $\Phi(\lambda) = (c\lambda)^\alpha$ for some $c > 0$ and $\lambda > 0$ (see, e.g. Sato (1999)).
- L is the regulator at 0, which is the magnitude of the minimal displacement that can keep X always nonnegative.

- Similarly as before , the First passage time of X is defined as

$$T_l = \inf\{t \geq 0;\ X_t \geq l\}, \quad l \in (x, \infty).$$

- Note that, since X is spectrally negative process, then $X_{T_y} = y$.

Theorem (BWY, Stoch. & Dyn. 2012)

The LT of the First passage time T_l *is given by*

$$\mathbf{E}_x\left[\exp\left(-\theta T_l\right)\right] = \frac{H^{\alpha}_{d\theta}(d^{-1/\alpha}c^{-1}x)}{H^{\alpha}_{d\theta}(d^{-1/\alpha}c^{-1}y)}, \quad 0 \le x < l,$$

where $d = (\alpha\gamma)^{-1}$ *and* $H^{\alpha}_{\delta}(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+\delta)}{\Gamma(\alpha k+1)} x^{\alpha k}$ *with Gamma function* $\Gamma(u) = \int_0^{\infty} e^{-v} v^{u-1} \mathrm{d}v$ *for* $u > 0$.

Structural Approach in Credit Risk Modeling

- **Merton's Default Definition**: Assume that a firm market value follows a random dynamic $(X_t;\ t \geq 0)$, and $K > 0$ is a default barrier. Then the default time τ is defined by a random variable:

$$\tau = \begin{cases} T, & \text{if } X_T < K, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $T > 0$ is a maturity time.

- **Thus the Probability of Default (PD)** for the firm may be given by

$$p_T = \mathbf{P}(X_T < K).$$

Example

- Suppose that a firm asset price is driven by a GBM:

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \;\; X_0 > K,$$

with the return rate μ and the volatility σ.

- Then the PD can be figured out by

$$\begin{aligned} p_T &= \mathbf{P}\left(W_T < \log(L) - (\mu - \frac{\sigma^2}{2})T\right) \\ &= \Phi\left(\frac{\log(L) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \end{aligned}$$

where $L = K/X_0 \in (0, 1)$ is the initial leverage ratio.

- In the Structural Model, the standard way is to first assume a underlying asset dynamics $X = (X_t;\ t \geq 0)$, and then to set a default barrier $K > 0$. The default time τ is generally defined as a positive r.v.:

$$\tau = \inf\{t > 0;\ X_t < K\},$$

where $X_0 > K$ and $\inf \emptyset = +\infty$.

- The PD before T is then given by

$$p_T := \mathbf{P}\left(\tau < T\right) = \mathbf{P}\left(\inf_{0 \leq t \leq T} X_t < K\right).$$

Example

- Assume that the underlying asset price is driven by a Geometric BM.
- The PD before T can be expressed explicitly by

$$
\begin{aligned}
p_T &= \mathbf{P}\left(\inf_{0\le t\le T}[mt+\sigma W_t] < \log(L)\right) \\
&= \Phi\left(\frac{\log(L)-mT}{\sigma\sqrt{T}}\right) + L^{2m/\sigma^2}\Phi\left(\frac{\log(L)+mT}{\sigma\sqrt{T}}\right),
\end{aligned}
$$

where $m=\mu-\frac{1}{2}\sigma^2$.

- **REMARK:** In the structural approach in Credit Risk, the Default Time actually corresponds to the so-called First Passage Time for a stochastic process in the context of Advanced Probability Theory. Hence we could theoretically switch the discussions on the Default Issues in the Credit Risk into the First Passage Issues of the underlying asset price (stochastic) process. In the modelings, we always assume that the market asset price dynamics is driven by a reflected O-U process.
- **O**n the other hand, for the application context, we might extend the default barrier K further to a time-varying barrier or even a random barrier (see the sections below).

Example

- Consider a GBM asset price dynamics, and
- Set the time-dependent default barrier by

$$K(t) = K \cdot e^{-k(T-t)}, \quad K, k > 0.$$

Then the PD can be calculated as follows

Example (Cont.)

$$
\begin{aligned}
p_T &= \mathbf{P}\left(\inf_{0\le t\le T}[(m-k)t+\sigma W_t] < \log(L) - kT\right) \\
&= \Phi\left(\frac{\log(L)-mT}{\sigma\sqrt{T}}\right) \\
&\quad + (Le^{-kT})^{2(m-k)/\sigma^2}\,\Phi\left(\frac{\log(L)+(m-2k)T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

Reduced-Form Approach

- In the Reduced Form Approach to Credit Risk, the central point is to take the default time τ as an exogenous random variable.
- Assume that there exists a non-negative process $(\lambda_t;\ t \geq 0)$ such that the default time can be formulated by

$$\mathbf{P}\left(\tau \in (t, t+\Delta t] | \tau > t\right) = \lambda_t \Delta t + o(\Delta t),$$

  for small $\Delta t > 0$.
- Here the non-negative process $(\lambda_t;\ t \geq 0)$ is called the default intensity process in the reduced form setting.

Example

- The default intensity λ_t is a deterministic function.
- Then the PD before T is given by

$$p_T = \mathbf{P}\left(\tau < T\right) = 1 - \exp\left(-\int_0^T \lambda_t \mathrm{d}t\right).$$

Example

- In Particular, an interesting parametric intensity is given by

$$\lambda_t := a_i, \quad \text{on } t \in [T_{i-1}, T_i),\ i = 1, 2, \ldots,$$

for some positive constants $a_i,\ T_i$. The advantage for this model is that those parameters might be calibrated from the market data.

The Conditional DP with Incomplete Information

- In this subsection, we consider, under the structural framework, the conditional default probability (CDP) with incomplete information.
- Usually, the complete information on the market price is unavailable. Specifically, we can only observe the market price at some discrete times, this can be interpreted as the quarterly provided reports on the asset evaluations of the firm (see, e.g. Duffie and Lando (2000)).
- The observed values also include some noises, this may be caused by a noisy accounting report of assets.

- **I**f let τ denote the default time of a firm, and let $\mathcal{F}_t$ denote the information flow till the time t.
- **D**efine the conditional survival probability (CSP) by

$$\mathbf{P}(\tau > t|\mathcal{F}_s), \qquad 0 < s < t.$$

Then we may have

- **Q**uestion 1: How to describe the default time τ ?
- **Q**uestion 2: How much information $\mathcal{F}_s$ is available for us at time $s > 0$?

- We use the first passage approach to define the default time:

$$\tau = \inf\{t > 0;\ X_t < d\}, \qquad X_0 > d,$$

which means that when the price crosses some pre-specified level $d \in [0, b]$, default occurs.
- We need $D_t := \mathbf{1}_{\{\tau \le t\}}$, and we call $D = (D_t;\ t \ge 0)$ the default indicator process.
- We assume, under the physical (statistical) measure, the market price follows a reflected O-U process

$$\mathrm{d}X_t = (\mu - \alpha X_t)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t - \mathrm{d}U_t, \ \ X_0 \in [0, b].$$

- Let $0 \leq t_1 < t_2 < \cdots < t_n < \cdots$ be a sequence of deterministic observed times. For each $t > 0$ fixed, define $n_t := \max\{j;\ t_j \leq t\}$.
- Denote the observed price at time t_i by $Y_{t_i} := X_{t_i} + \xi_{t_i}$, where $\xi = (\xi_t;\ t \geq 0)$ is an extra noisy source independent of X.
- The partial information is $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \subset \mathcal{G}$, where

$$\mathcal{F}_t = \sigma\left(\{Y_{t_1}, \cdots, Y_{t_{n_t}}\}\right) \vee \sigma\left(\{D_u;\ 0 \leq u \leq t\}\right).$$

- Introduce the Conditional Survival Probability. For each $(t, s) \in \mathbf{R}^2$ with $t > s$, define

$$\ell(s, t, Y_s) = \mathbf{P}(\tau > t | \mathcal{F}_s).$$

- We could get the explicit expression for the CSP $\ell(s, t, Y_s)$ for the case $t > s$ and $s = t_i$ with $i = 1, 2, \ldots$. We will consider the cases of single observation and multiple observations separately.
[Note: Here we only depict the single observation case, i.e., $s = t_1$].

Theorem (BWY, Quant. Finance, 2011.)

If $s = t_1$ *and* $t > s$. *Then CSP is determined by*

$$\ell(s,t,Y_s) = \frac{\int_d^b \mathbf{P}_u(\tau > t-s)h(\mathrm{d}u, Y_s, s)}{\int_d^b h(\mathrm{d}u, Y_s, s)},$$

Theorem (Cont.)

where

$$\frac{h(\mathrm{d}u, y, s)}{\mathrm{d}u} = \frac{\mathcal{F}_{\xi}(s; y - \mathrm{d}u)}{\mathcal{F}_{Y}(s; \mathrm{d}u)} \times \left[p(s; v, u) - \int_0^s p(s-r; d, u) \mathbf{P}_v(\tau \in \mathrm{d}r) \right]$$

and $p(\cdot;\cdot,\cdot)$ *is the transition density of ROU process.*

$$\mathcal{F}_X(t; \mathrm{d}x) := \mathbf{P}_v(X_t \in \mathrm{d}x), \quad t \geq 0.$$

Counterparty Risk and Random Recovery

- Among all sorts of the credit derivatives, CDS might be one of most popular credit instruments. In a CDS contract, for the protecting buyer the credit quality of the counterparty risk is very important, due to the over the counter trade.
- We take the counterparty risk account into the pricing of a CDS and we will adopt an intensity-based reduced form model, in which the default intensity process of the counterparty and the reference credit are modulated by the credit states of the firms as well.

- Two Markov chains are employed to describe the credit state processes. We can setup a model where the default correlation between the counterpart and the reference is described through the Markov chains.
- Eventually we can get a semi-explicit formula for the pricing of CDS with counterparty risk. See, e.g.

Counterparty risk for Credit Default Swap with States Related Default Intensity Processes, (Tang-Wang-Zhou). *Inter. J. Theo. Appl. Finance* **14**(8): (2011), 1335-1353.

- We propose a reflected recovery rate model and consider the pricings of defaultable bond and of CDS, under a flexible correlation structure among the short rate, the default intensity and the loss given default.
- The eigenfunction expansion technique is adopted to derive the explicit pricing formulae and two solvable examples are given.
- We also calibrate the model by fitting a small set of IBM CDS data.

- Empirical studies suggest that: First, the data support the recovery of a fraction of the pre-default debt value; Second, the default intensity possesses a relatively strong positive correlation with CDS spread, while the recovery rate does not possess this property. For details see

Modeling the recovery rate by reflected diffusions and related pricing (with L. Bo and X. Yang), (2012).

Some Works in Applications

Some of our recent works in applications.

- First passage times of Ornstein-Uhlenbeck processes over random jump boundaries (with L.Bo and X. Yang). *J. Appl. Probab.* **48**(3): (2011).
- Maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes (with L. Bo, X. Yang and G. Zhang). *J. Stat. Plann. Inference* **141**:1 (2011).
- An optimal portfolio problem in a defaultable market (with L. Bo, X. Yang). *Adv. Appl. Probab.* **42**:3 (2010), 689-705.

- Lévy risk model with two-sided jumps and a barrier dividend strategy (with L. Bo, R. Song, D. Tang and X. Yang). *Insurance: Math. Econ.* **50**(2): (2012), 280-291.
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The End

Thank you !