Generalized time-changes and Continuous-State Branching Processes with Immigration

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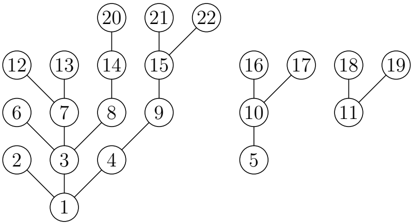
Breadth-first walks and profiles



- χ_i: # children of individual i.

- y_n: # immigrants up to generation n.
- c_n: # individuals up to generation n.

$$c_n = c_0 + y_n + \chi_1 + \cdots + \chi_{c_{n-1}}$$

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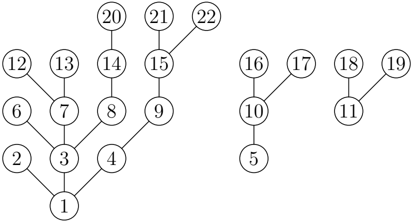


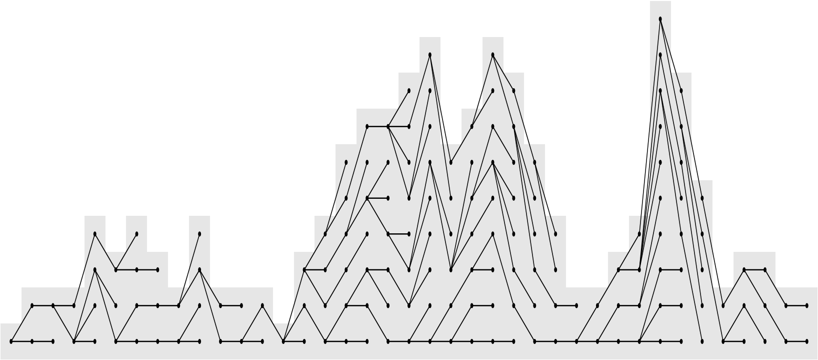
- χ_i: # children of individual i.
- $x_n = \chi_1 + \cdots + \chi_n - n$.
- y_n: # immigrants up to generation n.
- c_n: # individuals up to generation n.
- z_n: # individuals comprising generation n.

$$
\begin{aligned}
c_n &= c_0 + y_n + \chi_1 + \cdots + \chi_{c_{n-1}} \\
&= z_0 + \cdots + z_n \\
z_n &= c_0 + x_{c_{n-1}} + y_n
\end{aligned}
$$

A representation of GWI processes

- μ reproduction law, ν immigration law.
- $\tilde{\mu}(k) = \mu(k+1)$.
- X a random walk with step distribution $\tilde{\mu}$.
- Y an independent random walk with step distribution ν.
- $Z_0 = k$ and for $n \geq 1$:

$$Z_n = k + X_{Z_0 + \cdots + Z_{n-1}} + Y_n.$$



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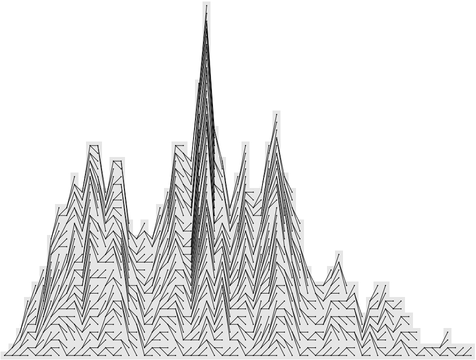

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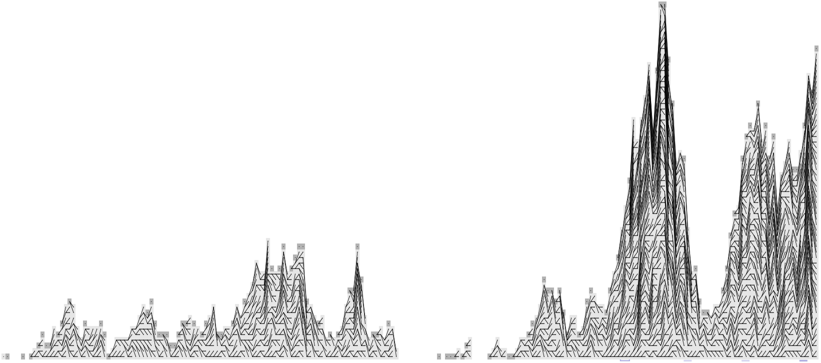

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Proposed extension:

X a SPLP, Y an independent subordinator and $x \geq 0$

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t.$$

CB Processes with Immigration

Definition

- Z: Markov process on $[0, \infty)$ with regular paths.
- Associated laws: $\mathbb{P}_x, x \geq 0$.
- Z is a CBI process if

$$\mathbb{E}_x\left(e^{-\lambda Z_t}\right) = e^{-x u_t(\lambda) - v_t(\lambda)}.$$

- Kawazu-Watanabe, 1971:

$$\frac{\partial}{\partial t} u_t(\lambda) = -\psi \circ u_t(\lambda) \quad \frac{\partial}{\partial t} v_t(\lambda) = \varphi \circ u_t(\lambda)\,.$$

- Branching property: $\mathbb{P}_x^{\psi,\varphi} * \mathbb{P}_y^{\psi,\tilde{\varphi}} = \mathbb{P}_{x+y}^{\psi,\varphi+\tilde{\varphi}}$.

An initial value problem

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_t$$

Initial Value Problem:

Let f, g be càdlàg with $\Delta f \geq 0$, g increasing and $f(0) + g(0) \geq 0$.
A function c solves IVP(f, g) if

$$c'_+ = f \circ c + g \quad \text{and} \quad c_0 = 0.$$

- f: reproduction function
- g: immigration function
- c: cumulative population
- $h = c'_+$: profile

Obvious problems

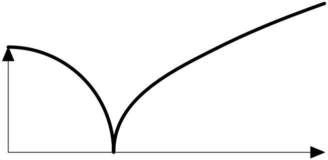
1. Existence?
2. Uniqueness?

The Lamperti transformation, existence, and uniqueness

$$c'_+ = f \circ c. \qquad i = c^{-1} \quad i' = \frac{1}{f \circ c \circ i} = \frac{1}{f}!!!$$

Problem: $f(x) = \sqrt{|1-x|}$. Then there are many solutions: their derivatives are

$$\left(\frac{2-x}{2}\right)^+ \quad \text{and} \quad \begin{cases} \frac{2-x}{2} & x < 2 \\ 0 & 2 \le x \le 2+l \\ \frac{x-2-l}{2} & x \ge 2+l \end{cases}$$



Existence and uniqueness for IVP(f, g)

$$c'_+ = f \circ c + g$$

Theorem

Let f, g be càdlàg $\Delta f \geq 0$, g increasing, $f(0) + g(0) \geq 0$. There exists a non-decreasing c which satisfies IVP(f, g). If g is strictly increasing the solution is unique.

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Corollary

Let X be an α stable spectrally positive Lévy process ($\psi(\lambda) = c\lambda^\alpha$) and Y and independent strictly increasing stochastic process. Then weak existence and uniqueness holds for

$$Z_t = x + \int_0^t |Z_s|^{1/\alpha} \, dX_s + Y_t.$$

In particular, weak existence and uniqueness holds for Lambert's SDE: Y is an $\alpha - 1$-stable subordinator whose Laplace exponent is ψ'.

The Lamperti type representation of CBI

Theorem

Let X be a SPLP and Y and independent subordinator. For any $x \geq 0$ there exists a unique solution to

$$Z_t = x + X_{\int_0^t Z_s \, ds} + Y_s.$$

It is a CBI whose branching and immigration mechanisms are the Laplace exponents of X and Y.

Applications of the representation theorem

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t$$

1. Given ψ and φ, there exists a CBI(ψ, φ).
2. A CBI(ψ, φ) does not jump downwards.
3. A CBI(ψ, φ) can reach ∞ by a jump if and only if $\psi(0) > 0$ or $\varphi(0) > 0$.
4. A CBI(ψ, φ) can reach ∞ continuously (explodes) if and only if $\int^\infty 1/\psi < \infty$.
5. Let $x > 0$, $\tilde{\varphi} = \psi^{-1}$ and

   $$f(t) = \frac{\log|\log t|}{\tilde{\varphi}(t^{-1}\log|\log t|)}.$$

   Then there exists $c \neq 0$ such that

   $$\liminf_{t \to 0} \frac{Z_t - x}{f(xt)} = c.$$

Discretization: Euler's method

- $\sigma_n \to 0$
- $t_i = i\sigma_n$, $i \geq 0$.
- $c^n(0) = 0$.
- $c^n(t) = c^\sigma(t_{i-1}) + (t - t_{i-1})\,[f_n \circ c^n(t_{i-1}) + g_n(t_{i-1})]^+$.
- If $\sigma_n = 0$, we let c^n be any solution to IVP(f_n, g_n).

Stability theorem

If there exists a unique continuous function c such that

$$\int_s^t f_- \circ c(r) + g(r)\ dr \leq c(t) - c(s) \leq \int_s^t f \circ c(r) + g(r)\ dr$$

and $f_n \to f$ and $g_n \to g$ then $c^n \to c$. Furthermore, if $f \circ c$ and g do not jump at the same time then $D_+ c^n \to D_+ c$.

Weak continuity of CBI laws

Corollary

Let $\psi_n \to \psi$ and $\varphi_n \to \varphi$ pointwise and $x_n \to x$. Then $\mathrm{CBI}_{x_n}(\psi_n, \varphi_n) \to \mathrm{CBI}_x(\psi, \varphi)$.

Limit theorems for GWI processes

Corollary

- X^n random walk with step distribution $\mu^n_{k+1}, k \geq -1$.
- Y^n random walk with step distribution $\nu^n_k, k \geq 0$.
- $X^n_{c_n}/n \to \mu$ (μ is sP ID with Laplace exponent ψ).
- $Y^n_{d_n}/n \to \nu$ (ν corresponds to a subordinator with Laplace exponent φ).
- Z^n is GW(μ^n, ν^n), $Z^n_0 = k_n$
- $\frac{k_n d_{\frac{k_n}{x}}}{x c_{\frac{k_m}{x}}} \to c \in [0, \infty)$
- $\frac{x}{k_n} Z^n_{d_{\frac{k_n}{x}} t}$ converges weakly to CBI$_x(c\psi, \varphi)$.

Limit theorems for Conditioned GW processes

Theorem

- μ critical and aperiodic offspring law.
- S random walk with step distribution $\mu_{k+1}, k \geq -1$.
- S_n/a_n converges weakly to (sp) stable law of index $\alpha \in (1,2]$.
- Z^{n,k_n} with law $\mathrm{GW}_{k_n}(\mu)$ and conditioned on

$$\sum_i Z_i^{n,k_n} = n.$$

- $k_n/a_n \to l > 0$.
- F^l: first passage bridge of α stable spLp.

Then

$$\left(\frac{a_n}{n} Z_{nt}^{n,k_n}\right)_{t\geq 0} \to \text{solution of } \mathrm{IVP}\left(F^l, 0\right).$$