

# Exponential convergence of Markov Processes

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# 1. Introduction

- $(M, \mathcal{B}, m)$  : a measure space with  $m(M) = 1$
- $\{T_t\}$  : a Markovian semigroup in  $L^2(m)$

We assume

- $\{T_t^*\}$  : a Markovian semigroup in  $L^2(m)$
- $T_t \mathbf{1} = T_t^* \mathbf{1} = \mathbf{1}$

Then  $\{T_t\}$  and  $\{T_t^*\}$  define semigroups in  $L^p(m)$  ( $1 \leq p \leq \infty$ ). For  $f \in L^1$ , we denote

$$\langle f \rangle = \int_M f \, dm.$$

We are interested in the following ergodicity:

$$T_t f \rightarrow \langle f \rangle \quad \text{as } t \rightarrow \infty$$

To be precise, define the index  $\gamma_{p \rightarrow q}$  by

$$(1) \quad \gamma_{p \rightarrow q} = - \overline{\lim} \frac{1}{t} \log \|T_t - m\|_{p \rightarrow q}.$$

Here

- $m$  : an operator  $f \mapsto m(f) = \int_X f \, dm$
- $\|\cdot\|_{p \rightarrow q}$  : operator norm from  $L^p$  to  $L^q$

We are interested in how  $\gamma_{p \rightarrow q}$  depends on  $p$  and  $q$ .

From the Riesz-Thorin interpolation theorem, we have

$$s \mapsto \gamma_{1/s \rightarrow 1/s}$$

is concave. So if the semigroup is symmetric,  $\gamma_{2 \rightarrow 2}$  is the largest.

## 2. Hypercontractivity and the exponential convergence

### Hyperboundedness

$\{T_t\}$  is called **hyperbounded** if there exist  $K > 0$ ,  $r \in (2, \infty)$  and  $C \geq 1$  such that

$$\|T_K f\|_r \leq C \|f\|_2, \quad \forall f \in L^2(m).$$

**Theorem 1.** The followings are equivalent to each other:

- (1)  $\{T_t\}$  is hyperbounded.
- (2)  $\gamma_{p \rightarrow q} \geq 0$  for some  $1 < p < q < \infty$ .
- (3)  $\gamma_{p \rightarrow q} = \gamma_{2 \rightarrow 2}$  for all  $p, q \in (1, \infty)$ .

## Hypercontractivity

$\{T_t\}$  is called **hypercontractive** if there exist  $T > 0$  and  $r \in (2, \infty)$  such that

$$(2) \quad \|T_T f\|_r \leq \|f\|_2, \quad \forall f \in L^2(m).$$

**Proposition 2.** Under (2), we have

$$\|T_T f - \langle f \rangle\|_2 \leq (r - 1)^{-1/2} \|f\|_2, \quad \forall f \in L^2(m).$$

Furthermore, for any  $t \geq 0$ , we have

$$\|T_t f - \langle f \rangle\|_2 \leq \sqrt{r - 1} \exp\left\{-\frac{t}{T} \log \sqrt{r - 1}\right\} \|f\|_2, \quad \forall f \in L^2(m).$$



**Proposition 3.** Let  $r > 2$ . Suppose that there exist positive constants  $T_0, T_1$  such that

$$M_0 := \|T_{T_0}\|_{2 \rightarrow r} < \infty$$

$$\rho := \sup\{\|T_{T_1}f - \langle f \rangle\|_2 / \|f\|_2 : f \in L^2(m) \setminus \{0\}\} < 1.$$

Then the semigroup  $\{T_t\}$  is hypercontractive.

**Theorem 4.** The followings are equivalent to each other:

- (1)  $\{T_t\}$  is hypercontractive.
- (2)  $\gamma_{p \rightarrow q} > 0$  for some  $1 < p < q < \infty$ .
- (3)  $\gamma_{p \rightarrow q} = \gamma_{2 \rightarrow 2} > 0$  for all  $p, q \in (1, \infty)$ .

### 3. Sufficient condition for independence of $L^p$ -spectrum

#### Normal operator

- $\mathfrak{A}$ : the generator of  $\{T_t\}$
- $\mathfrak{A}^*$ : the generator of  $\{T_t^*\}$

We assume that  $\mathfrak{A}$  is normal, i.e.,

$$\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$$

Then  $\mathfrak{A}$  has the spectral decomposition:

$$-\mathfrak{A} = \int_{\mathbb{C}} \lambda dE_{\lambda}$$

## Multiplier

For any bounded function  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ , define  $\phi(-\mathfrak{A})$  by

$$\phi(-\mathfrak{A}) = \int_{\mathbb{C}} \phi(\lambda) dE_{\lambda}.$$

**Theorem 5.** Assume that  $\{T_t\}$  is hyperbounded.

If  $\phi(\lambda)$  is expressed as

$$\phi(\lambda) = h(1/\lambda)$$

for a bounded function  $h$  on  $\mathbb{C}$  which is analytic near 0. Then  $\phi(-\mathfrak{A})$  is bounded in  $L^p(m)$ .

Using this theorem, we can show that the boundedness of the resolvent is independent of  $p$ .

**Theorem 6.** Assume  $\mathfrak{A}$  is normal. Then  $\sigma(\mathfrak{A}_p)$ , the spectrum of  $\mathfrak{A}_p$ , is independent of  $p$ .

## 4. Example of $L^p$ -spectrum that depend on $p$

We give an example that the spectrum depends on  $p$ .

- $M = [0, \infty)$
- $m(dx) = \nu(dx) = e^{-x} dx$
- The Dirichlet form in  $L^2(\nu)$ :

$$\mathcal{E}(f, g) = \int_{[0, \infty)} f'(x)g'(x)\nu(dx)$$

with domain

$$\text{Dom}(\mathcal{E}) = \{f \in L^2(\nu); f \text{ is absolutely continuous and } f' \in L^2(\nu)\}.$$

- The generator:

$$\mathfrak{A} = \frac{d^2}{dx^2} - \frac{d}{dx}$$

with domain

$$\text{Dom}(\mathfrak{A}) = \{f \in \text{Dom}(\mathcal{E}); f' \text{ is absolutely continuous and } f'' \in L^2(\nu) \text{ with } f'(0) = 0\}.$$

To see the spectrum of  $\mathfrak{A}$ , we introduce the following unitary transformation

$$If(x) = e^{-x/2} f(x)$$

and

$$I^{-1}f(x) = e^{x/2} f(x).$$

We note

$$I \circ \mathfrak{A} \circ I^{-1} f = -\frac{1}{4} f + \frac{d^2 f}{dx^2}$$

i.e., we have the following commutative diagram:

$$\begin{array}{ccc} L^2(\nu) & \xrightarrow{\mathfrak{A}} & L^2(\nu) \\ I \downarrow & & \downarrow I \\ L^2(dx) & \xrightarrow{\frac{d^2}{dx^2} - \frac{1}{4}} & L^2(dx) \end{array}$$

The boundary condition is involved as follows:

$$\begin{array}{c} \mathfrak{A} \quad \text{with } f'(0) = 0 \\ I \downarrow \\ \frac{d^2}{dx^2} - \frac{1}{4} \quad \text{with } \frac{1}{2} f(0) + f'(0) = 0 \end{array}$$

The corresponding Dirichlet form  $\hat{\mathcal{E}}$ :

$$\begin{aligned}\hat{\mathcal{E}}(f, g) &= \mathcal{E}(I^{-1}f, I^{-1}g) \\ &= \int_0^\infty f'g' dx + \frac{1}{4} \int_0^\infty fg dx - \frac{1}{2}f(0)g(0).\end{aligned}$$

We can show that  $\hat{\mathcal{E}}$  is a compact perturbation of

$$\hat{\mathcal{E}}^{(0)}(f, g) = \int_0^\infty f'g' dx + \frac{1}{4} \int_0^\infty fg dx.$$

It is easy to see that spectrum corresponding to  $\hat{\mathcal{E}}^{(0)}(f, g)$  is

$$\left[\frac{1}{4}, \infty\right)$$



Now setting

$$A = \frac{d^2}{dx^2} - \frac{1}{4}$$

with  $\text{Dom}(A) = \{f, f'' \in L^2([0, \infty)) \text{ with } \frac{1}{2}f(0) + f'(0) = 0\}$ ,  
we have

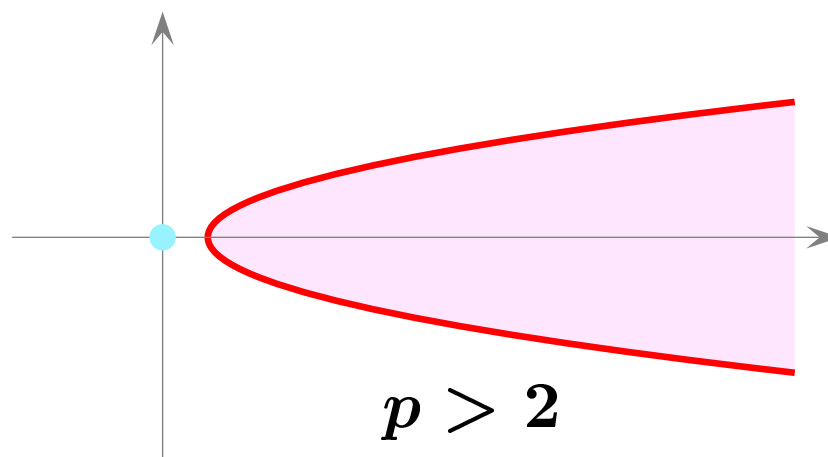
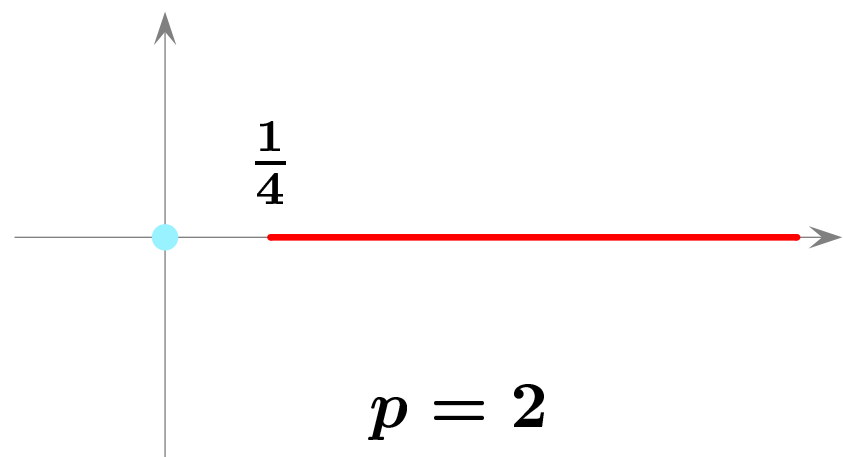
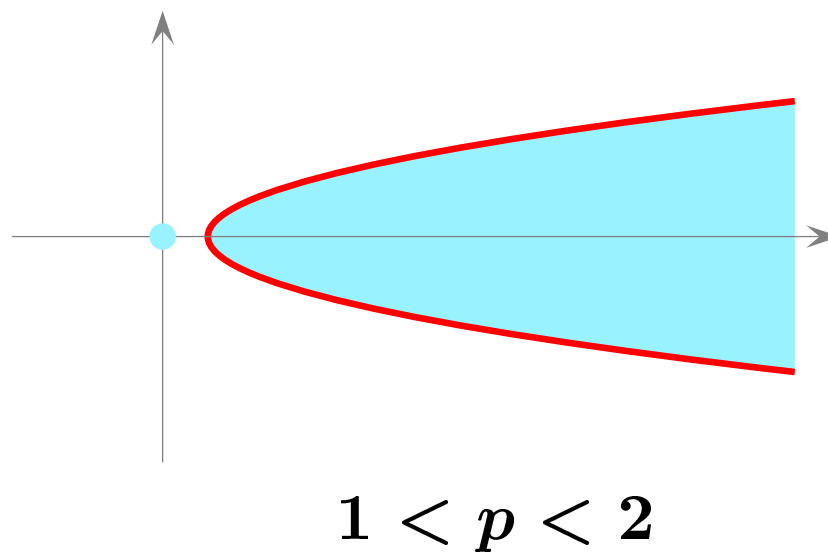
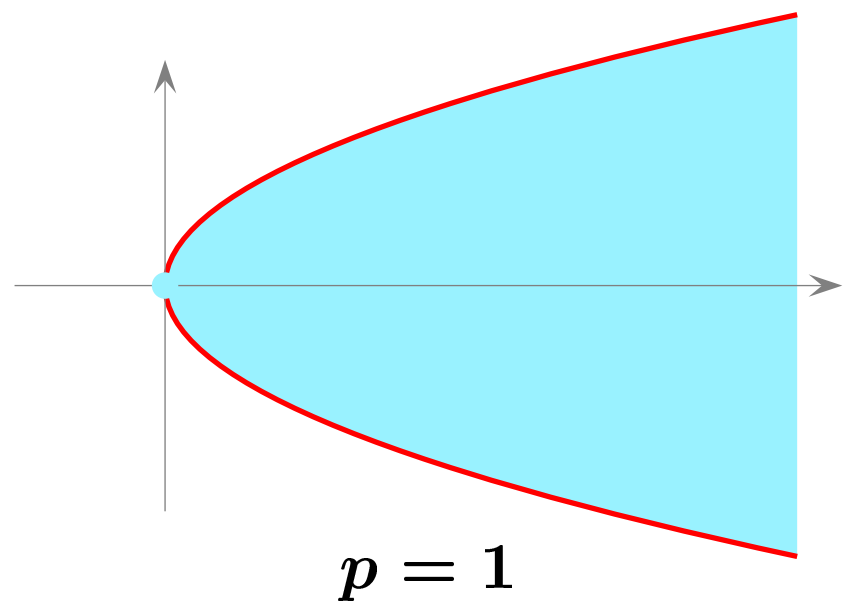
$$\sigma(-A) = \{0\} \cup \left[\frac{1}{4}, \infty\right).$$

Now, by the unitary equivalence,

**Theorem 7.** We have

$$\sigma(-\mathfrak{A}) = \{0\} \cup \left[\frac{1}{4}, \infty\right).$$

Now we proceed to the  $L^p$ -spectrum. The result is



First we discuss the case  $1 \leq p < 2$ . We use the same mapping  $I f(x) = e^{-x/2} f(x)$  in as

$$I: L^p(\nu) \longrightarrow L^p(\tilde{\nu})$$

where

$$\tilde{\nu}(dx) = e^{(p/2-1)x} dx.$$

Then  $I$  gives an isometry between  $L^p(\nu)$  and  $L^p(\tilde{\nu})$ . Similarly as before, setting  $\tilde{\mathfrak{A}} = I \circ \mathfrak{A} \circ I^{-1}$ , we have

$$\tilde{\mathfrak{A}} f = \frac{d^2 f}{dx^2} - \frac{1}{4} f$$

with the boundary condition

$$f'(0) + \frac{1}{2} f(0) = 0.$$

**Proposition 8.** For  $1 \leq p < 2$ , we have

$$\sigma_p(-\mathfrak{A}) = \{0\} \cup \left\{ x + iy; x, y \in \mathbb{R}, y^2 < \left(\frac{2}{p} - 1\right)^2 \left(x - \frac{p-1}{p^2}\right) \right\}$$

*Proof.* We solve the following differential equation:

$$\begin{cases} -u'' + \frac{1}{4}u = \lambda u, \\ u'(0) + \frac{1}{2}u(0) = 0. \end{cases}$$

The solution is given by

$$\begin{cases} u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}}, \\ C_1 \left(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}\right) + C_2 \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right) = 0. \end{cases}$$

By checking the integrability, we get the desired result. □

**Proposition 9.** For  $1 \leq p < 2$ , we have

$$\rho(-\mathfrak{A}) \supseteq \left\{ x + iy; x, y \in \mathbb{R}, y^2 > \left( \frac{2}{p} - 1 \right)^2 \left( x - \frac{p-1}{p^2} \right) \right\} \setminus \{0\}$$

*Proof.* For  $\lambda \in \{z \in \mathbb{C}; \Re \sqrt{-z + 1/4} > \frac{1}{p} - \frac{1}{2}\}$ , define

$$\begin{aligned} \phi_\lambda(x) = & \left( \frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}} \right) e^{x\sqrt{-\lambda + 1/4}} \\ & - \left( \frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}} \right) e^{-x\sqrt{-\lambda + 1/4}}, \end{aligned}$$

$$\psi_\lambda(x) = e^{-x\sqrt{-\lambda + 1/4}},$$

$$W_\lambda = -2\sqrt{-\lambda + \frac{1}{4}} \left( \frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}} \right).$$

Further, define  $g_\lambda : [0, \infty) \times [0, \infty) \rightarrow \mathbb{C}$  を

$$g_\lambda(x, y) = \begin{cases} \frac{1}{W_\lambda} \phi_\lambda(x) \psi_\lambda(y), & x \leq y, \\ \frac{1}{W_\lambda} \phi_\lambda(y) \psi_\lambda(x), & x \geq y. \end{cases}$$

The possible Green operator  $G_\lambda$  is given by

$$G_\lambda f(x) = \int_0^\infty g_\lambda(x, y) f(y) dy.$$

For  $f \in C_0^\infty([0, \infty) \rightarrow \mathbb{C})$ , we have

$$(\lambda + \tilde{\mathfrak{A}})G_\lambda f = f, \quad \frac{1}{2}G_\lambda f(0) + (G_\lambda f)'(0) = 0.$$

Now it suffices to show that  $G_\lambda$  is a bounded operator. □

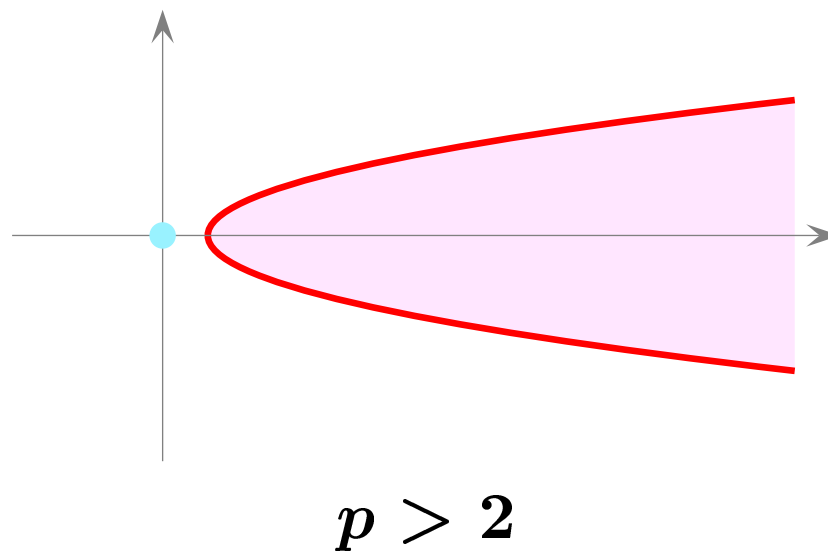
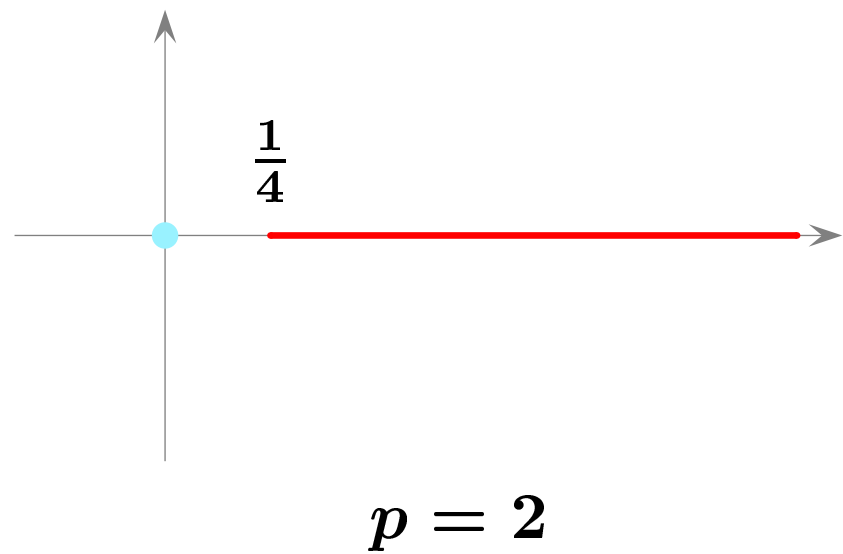
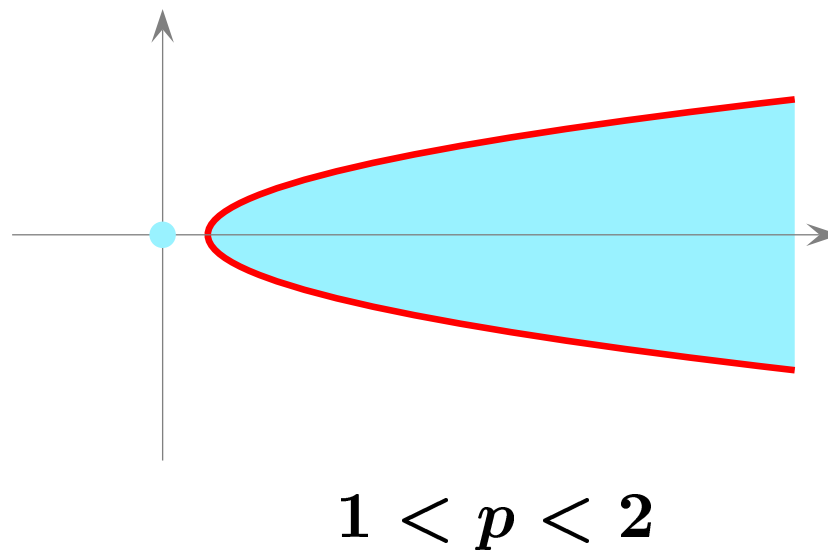
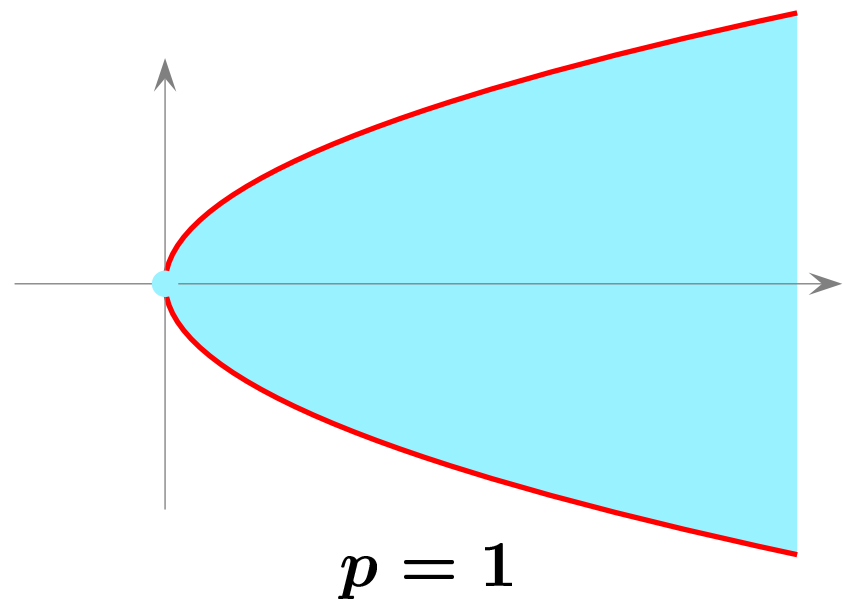
We can summarize as follows:

**Theorem 10.** For  $1 \leq p < 2$ , we have

- (i)  $\sigma_p(-\mathfrak{A}) = \{0\} \cup \{x + iy; x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$
- (ii)  $\sigma_c(-\mathfrak{A}) = \{x + iy; x, y \in \mathbb{R}, y^2 = (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$
- (iii)  $\rho(-\mathfrak{A}) = \{x + iy; x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

**Theorem 11.** For  $p > 2$ , we have

- (i)  $\sigma_p(-\mathfrak{A}) = \{0\}$
- (ii)  $\sigma_r(-\mathfrak{A}) = \{x + iy; x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$
- (iii)  $\sigma_c(-\mathfrak{A}) = \{x + iy; x, y \in \mathbb{R}, y^2 = (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$
- (iv)  $\rho(-\mathfrak{A}) = \{x + iy; x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$





By noting that

$$\inf\{\Re\lambda; \lambda \in \sigma(-\mathfrak{A}) \setminus \{0\}\} = -\lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t - m\|$$

we have

**Theorem 12.** For  $1 \leq p < \infty$

$$\gamma_{p \rightarrow p} = \frac{p-1}{p^2}.$$

Thanks !