The Resolvents of Drift-accelerated Diffusions

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We consider $X^{(c)}(t)$:

$$dX^{(c)}(t) = cb(X^{(c)}(t))dt + dB(t). \tag{1.1}$$

$B(t)$ is the d-dim Brownian motion.

- $b(\cdot)$ is a smooth vector field with period 1.
  b is divergence free. That is,

  $$div(b) = 0. \tag{1.2}$$

- $X^{(c)}(t)$ is a diffusion process on d-dim torus.
- $X^{(c)}(t)$ has the Lebesgue measure as the invariance measure.
- We are interested in the behaviors of $X^{(c)}(t)$ as $c \to \infty$.

1. Introduction-2

We denote $\phi(t)$ the trajectory of the dynamical system:

$$\frac{d\phi}{dt} = b(\phi(t)). \tag{1.3}$$

When c is large, we see the following picture for (1.1).

- In a short time interval after t, $X^{(c)}$ traces a curve containing $X^{(c)}(t)$ following (1.3).
- Due to the noise induced by $B(\cdot)$, we will have a random movement of curve of (1.3).

Question:

- How to describe this rigorously?
- Why is this interesting?

1. Introduction-3

This is a particular example of a more general class of diffusion processes,

$$dX^{(c)}(t) = (-\nabla U(X^{(c)}(t)) + cb(X^{(c)}(t))dt + dB(t),$$

with U, b periodic and satisfying

$$div(b\exp(-2U)) = 0,$$

such that they have μ as the invariance measure,

$$d\mu = \frac{1}{Z}\exp(-2U(x))dx.$$

- Such diffusion processes appear in MCMC(Markov Chain Monte Carlo).
- One chooses particular b to simulate μ.
- We may also consider μ on R^d and we do not assume periodicity of U.
- How good is the approximation of μ using $X^{(c)}(t)$? How to choose a better b?

Why $b(\cdot) \neq 0$ and c is large?

(I) For $b \neq 0$, as $t \to \infty$, $X^{(c)}(t)$ converges to the equilibrium faster when comparing with $X^{(0)}(t)$.
This explains why the case $b \neq 0$ is interesting.

(IIa) For $b \neq 0$ and $c \neq 0$, the diffusion $X^{(c)}(t)$ is nonreversible.
Calculation of the convergence rate of $X^{(c)}(t)$ to the equilibrium is in general impossible.

(IIb) For $b \neq 0$, as $c \to \infty$, it is possible to give an expression of the convergence rate of $X^{(c)}(t)$ asympototically.
This explains why to study the problem as $c \to \infty$ is interesting.

2. Convergence Rate-1

We assume $b(\cdot) \neq 0$.

- We denote
$$T_t^{(c)} f(x) = E_x[f(X^{(c)}(t))],$$
$$L^{(c)} f(x) = \frac{1}{2}\Delta f(x) + cb(x)\nabla f(x).$$
- We consider the largest ρ (denoted as $\rho(c)$) such that
$$\int |T_t^{(c)} f(x)|^2 dx \leq c_f \exp(-\rho t)$$
for large t and f satisfying
$$\int f(x)dx = 0, \quad \int |f(x)|^2 dx < \infty.$$

2. Convergence Rate-2

- We have

$$\rho(c) = \inf\{-Re(\rho); \rho \neq 0 \text{ is in the spectrum of } L^{(c)} \}.$$

  $\rho(c)$ is also called spectral gap
  This is the gap between 0 and the rest of spectrum
- $\rho(c)$ is used to measure the convergence rate of $X^{(c)}(t)$ to the equilibrium.

$$\int_{\mathbf{T}} |T_t^{(c)} f(x) - \pi(f)|^2 dx \leq c_f \exp(-\rho(c)t).$$

  Here

$$\pi(f) = \int_{\mathbf{T}} f(x) dx.$$

2. Convergence Rate-3

- The spectral gap for a self-adjoint operator can be expressed by a variational form.
- This is the case for $c = 0$.

$$\rho(0) = 2\pi^2 = \inf\{\tfrac{1}{2}\int_{\mathbf{T}} |\nabla f(x)|^2 dx; \int_{\mathbf{T}} f(x)dx = 0, \tfrac{1}{2}\int_{\mathbf{T}} f(x)^2 dx = 1\}.$$

- For $c \neq 0$, we can not have such expression for $\rho(c)$. This causes difficulty to calculate $\rho(c)$ and the difference $\rho(c) - \rho(0)$.
- However, we always have $\rho(c) \geq \rho(0)$.

$\rho(c) \geq \rho(0)$**:**

Here is a simple argument.
We assume $\pi(f) = 0$. Consider

$$\begin{aligned} \frac{d}{dt}\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx &= 2\int_{\mathbf{T}} T_t^{(c)} f(x) L^{(c)} T_t^{(c)} f(x) dx \\ &= -\int_{\mathbf{T}} \nabla T_t^{(c)} f(x) \nabla T_t^{(c)} f(x) dx \\ &\leq -2\rho(0)\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx. \end{aligned}$$

Then

$$\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx \leq \exp(-2\rho(0)) \int_{\mathbf{T}} |f(x)|^2 dx.$$

Therefore,

$$\rho(c) \geq \rho(0).$$

What is the gap of $\rho(c)$ and $\rho(0)$?

The following result is proved in Franke-Hwang-Pai-Sheu (2010).

Theorem 3.1.

$$\lim_{c\to\infty}\rho(c) = \inf\{\tfrac{1}{2}\int_{\mathbf{T}}|\nabla\psi(x)|^2 dx; \exists\mu, b\nabla\psi = i\mu\psi,\ \int_{\mathbf{T}}\psi(x)dx = 0, \int_{\mathbf{T}}|\psi(x)|^2 dx = 1\}.$$

In this expression, $\psi = \psi_1 + i\psi_2$, $i = \sqrt{-1}$.
Using such expression in some examples, $\rho(c)$ can be calculated approximately.

This result and our approach are both interesting.

A lower estimate (an easier part):

Theorem 3.2 We have

$$\begin{aligned} & \liminf_{c\to\infty} \rho(c) \\ (3.1) \qquad \geq \; & \inf\{\tfrac{1}{2}\int_{\mathbf{T}} |\nabla\psi(x)|^2 dx; \exists \mu, b\nabla\psi = i\mu\psi, \\ & \qquad \int_{\mathbf{T}} \psi(x)dx = 0, \int_{\mathbf{T}} |\psi(x)|^2 dx = 1\}. \end{aligned}$$

An upper estimate:

The upper estimate gives some interesting information about the spectrum of $L^{(c)}$.
To describe, we need some notations.

(3.2)
$$\begin{aligned} H^1 = \;& \{\psi = \psi_1 + i\psi_2; \psi, \nabla\psi \in L^2, \\ & \textstyle\int_{\mathbf{T}} \psi_1(x)dx = 0 = \int_{\mathbf{T}} \psi_2(x)dx = 0\}, \end{aligned}$$

(3.3)
$$H^1_\mu = \{\psi \in H^1; b\nabla\psi = i\mu\psi\},\ \mu \in R.$$

Using these notations, Theorem 3.1 can be stated

(3.4)
$$\begin{aligned} & \lim_{c\to\infty} \rho(c) \\ \geq\;& \inf\{\tfrac{1}{2}\textstyle\int |\nabla\psi_1|^2 + |\nabla\psi_2|^2; \int_{\mathbf{T}} \psi_1^2 + \psi_2^2 dx = 1, \\ & \exists\mu \text{ such that } \psi = \psi_1 + i\psi_2 \in H^1_\mu\}. \end{aligned}$$

Let $\mu \in R$. Assume H_μ^1 has nonzero element. Define

$$\rho_\mu = \inf\{\frac{\frac{1}{2}\int_{\mathbf{T}} |\nabla\psi(x)|^2 dx}{\int_{\mathbf{T}} |\psi(x)|^2 dx}; \psi \neq 0 \in H_\mu^1\}$$

Theorem 3.3. Let μ, ρ_μ be defined as above. Then for any $r > 0$, there is $c_0 = c_0(r)$ such that for all $c \geq c_0$, there is $-\bar{\rho} + i\bar{\mu}$ in the spectrum of $L^{(c)}$ such that

$$(\bar{\rho} - \rho_\mu)^2 + (\bar{\mu} - c\mu)^2 < r^2.$$

3. Limit of $\rho(c)$-5

- This theorem implies $L^{(c)}$ has an eigenvalue with large imaginary part near $c\mu$, if $\mu \neq 0$ and H_μ^1 contains nonzero elements.
- As another consequence of this theorem, we have

$$\limsup_{c\to\infty} \rho(c) \leq \rho_\mu,$$

if H_μ^1 contains nonzero elements. Together with (3.4), we have Theorem 3.1:

$$\lim_{c\to\infty} \rho(c) = \inf_\mu \{\rho_\mu\}$$

We show the convergence of $X^{(c)}(\cdot)$ in the following sense.

- We consider the resolvent of $X^{(c)}(t)$:

$$R_\lambda^{(c)} g(x) = \int_0^\infty e^{-\lambda t} T_t^{(c)} g(x) dt.$$

- We show the convergence of $R_\lambda^{(c)} g(x)$ as $c \to \infty$.
  The limit is denoted by $R_\lambda^* g(x)$.
- R_λ^* is a selfadjoint pseudo-resolvent on $L^2(T)$.

- The selfajointness of R^*_λ give a hope to use theory of Dirichlet form to construct a Markov process $X^*(t)$ with resolvent R^*_λ.
- On the other hand, the closure of the common range of R^*_λ in general is not the full space of $L^2(T)$.

On possible solution:

- Find a space X and a measure μ_X such that

$$\overline{R(R^*_\lambda)} \text{ is isometric to } L^2(X, \mu_X) \; .$$

$R(R^*_\lambda)$ is the range of R^*_λ.

- Then we can construct a Markov process $X^*(t)$ on X.
- The result is a random movement of the trajectory

$$\frac{d\phi(t)}{dt} = b(\phi(t)).$$

- Small random pertubation of Hamiltonian system is studied in several papers by Freidlin, Weber, Sower and others.

$$dX^{(\epsilon)}(t) = H_y(X^{(\epsilon)}(t), Y^{(\epsilon)}(t))dt$$
$$dY^{(\epsilon)}(t) = -H_x(X^{(\epsilon)}(t), Y^{(\epsilon)}(t))dt + \sqrt{\epsilon}dB(t),$$

$B(t)$ is a Brownian motion, ϵ is a small number,

$$H_x = \frac{\partial H}{\partial x}, \ H_y = \frac{\partial H}{\partial y}.$$

- Large time asymptotics: $\bar{X}^{(\epsilon)} = X^{(\epsilon)}(\frac{t}{\epsilon}), \bar{Y}^{(\epsilon)} = Y^{(\epsilon)}(\frac{t}{\epsilon})$

$$d\bar{X}^{(\epsilon)}(t) = \frac{1}{\epsilon}H_y(\bar{X}^{(\epsilon)}(t), \bar{Y}^{(\epsilon)}(t))dt$$
$$d\bar{Y}^{(\epsilon)}(t) = -\frac{1}{\epsilon}H_x(\bar{X}^{(\epsilon)}(t), \bar{Y}^{(\epsilon)}(t))dt + dB(t),$$

This is a system similar to (1.1).

- Only two dimensional space is considered where the geometry of the trajectoris is simpler.
- It may be possible to use our approach to generalize these results to the high dimensional spaces.

On d-dimensional torus, we consider

$$(5.1) \qquad \psi_{n_1}(x_1)\psi_{n_2}(x_2)\cdots\psi_{n_d}(x_d),$$

where $\psi_0(x) = \frac{1}{2\pi}$ and for $k \neq 0$,

$$\psi_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx), \text{ or } \frac{1}{\sqrt{\pi}}\cos(kx).$$

Each (5.1) is an eigenfunction of $\frac{1}{2}\Delta$. The eigenvalue is

$$-\frac{1}{2}(n_1^2 + n_2^2 + \cdots + n_d^2).$$

5. The Convergence of Resolvents-2

- We use $\phi_k, -\lambda_k$ to denote the collection of these eigenfunctions and eigenvalues.
- $\phi_k, k = 1, 2, \cdots$ is an orthonormal basis of $L^2(T)$. $\lambda_k \to \infty, k \to \infty$.
- We denote

$$H^m = \{\psi = \sum_{k=0}^{\infty} a_k \psi_k; \sum_{k=0}^{\infty} |a_k|^2 \lambda_k^m < \infty\}.$$

- For $\psi \in H$, $\Delta\psi \in L^2(T)$.
- If m is sufficiently large and $\psi \in H^m$, then ψ is continuous.

5. The Convergence of Resolvents-3

Theorem 5.1 For $g \in H^1$ and $\lambda > 0$, $R_\lambda^{(c)} g$ converges to a ψ_* in H^1 as $c \to \infty$.

(a) $B\psi_* = 0$, where we denote

$$B\psi = b\nabla\psi.$$

(b) We define

$$R_\lambda^* g = \psi_*.$$

Then R_λ^* is selfadjoint pseudo-resolvent.
The resolvent relation:

$$(\lambda_1 - \lambda_2) R_{\lambda_1}^* R_{\lambda_2}^* = R_{\lambda_2}^* - R_{\lambda_1}^*, \ \lambda_1, \lambda_2 > 0.$$

The selfadjointness:

$$< R_\lambda^* g_1, g_2 >=< g_1, R_\lambda^* g_2 > .$$

Here we denote

$$< f_1, f_2 >= \int_T f_1(x) f_2(x) dx.$$

Since $\psi_c = R_\lambda^{(c)} g$ satisfies

$$\frac{1}{2}\psi_c + cb\nabla\psi_c - \lambda\psi_c = -g,$$

we expect (a).

Theorem 5.2 For all $\lambda > 0$,

$$\overline{Range(R_\lambda^*)} = \overline{Ker(B) \cap H^1}.$$

We see the $\overline{Range(R_\lambda^*)}$ is not $L^2(T)$.

Let $\Phi_t(x)$ be the solution of

$$\frac{d\phi(t)}{dt} = b(\phi(t)),\ \phi(0) = x.$$

We define an equivalent relation on T as follows.
x, y are equivalent if for all $\epsilon > 0$ there is

$$z_1, z_2, \cdots, z_n,\ t_0, t_1, \cdots, t_n, s_1, s_2, \cdots, s_{n+1},$$

such that

$$d(\Phi_{t_0}(x), \Phi_{s_1}(z_1)) + d(\Phi_{t_1}(z_1), \Phi_{s_2}(z_2)) + \cdots d(\Phi_{t_n}(z_n), \Phi_{s_{n+1}}(y)) \leq \epsilon.$$

- The equivalent class containing x is denoted by $[x]$.
- X is the collection of all equivalent classes.

- We define $d([x],[y])$ equal to

  $$\inf\{d(\Phi_{t_0}(x),\Phi_{s_1}(z_1))+d(\Phi_{t_1}(z_1),\Phi_{s_2}(z_2))+\cdots d(\Phi_{t_n}(z_n),\Phi_{s_{n+1}}(y))\}$$

  over all possible n and

  $$z_1,z_2,\cdots,z_n,t_0,t_1,\cdots,t_n,s_1,s_2,\cdots,s_{n+1}.$$

- X is a metric space with metric $d([x],[y])$.
- We define $p_X : T \to X$, $p_X(x) = [x]$.
- Define $\mu = \pi o p_X^{-1}$ a probability measure on $(X, \mathcal{B}(X))$. π is the Lebesgue measure on T.

Theorem 6.1 The spaces $\overline{Ker(B) \cap H^1}$ and $L^2(X, \mu)$ are isometric.

- We define
$$\Sigma = \{p_X^{-1}(A); A \in \mathcal{B}(X)\}.$$
$$l_X^2 = \{\psi \in L^2(T) \colon \psi \text{ is } \Sigma\text{- measurable}\}.$$
- $P_l : L^2(T) \to l^2$ is the projection.
- For any Σ measurable ψ, there is a unique $\mathcal{B}(X)$ measurable ψ_X on X such that
$$\psi(x) = \psi_X(p_X(x)).$$
We write
$$\psi_X = R(\psi).$$
- The isometry in Theorem 6.1 is given by $\psi \to R(P_l(\psi))$.

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