BSDE driven by G-Brownian Motion

Shige Peng, Shandong University, China

Joint work with

Mingshang HU, Shaolin JI and Yongsheng SONG

The 8th Workshop on Markov Processes and Related Topics

16, July, 2012, Beijing Normal University

Uncertainty and Risk

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- new mathematical concept and calculation tool called nonlinear expectation theory which take the risk of model uncertainty (Knightian uncertainty) into account.
- Important: The existing results in probability theory, stochastic controls, mathematical finance, risk measures and risk controls are our rich sources.

Program of IMS Singapore University

- Title: Nonlinear Expectations, Stochastic Calculus under Knightian Uncertainty and Related Topics
- Time period (around): 3rd Jun to 12th Jul 2013
- (6 weeks, summer school and two workshops)
- Proposed by: M. Dai, H. Föllmer, J. Hinz (NUS)
  S. Peng, (SDU) J. Xia (AMSS, China) J. Zhang (USC)

Developments of research on uncertainty— an uncertain process?

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- W. Doeblin (1940, Pli cacheté)

Kolmogorov's Probability Space $(\Omega, \mathcal{F}, P)$
A fundamental and powerful theory and methodology to treat uncertainties

- A. Kolmogorov, Grundbegriffe der Wahrscheinlichkeitsrechnung. Springer, Berlin, 1933
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- Hilbert's 6th problem

- The von Neumann-Morgenstern utility axioms (1953) $E[U(X)]$; Theory of Games and Economic Behavior, Princeton;

Impacts in Economics & Math. Finance

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Frank H. Knight (1921) “Risk, Uncertainty and Profit”

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Knightian uncertainty

The prob. and distr. are unknown— "uncertainty of probability measures".

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- Hansen & Sargent: Robust control method.

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- Then define:

$$\mathbb{E}^g[X] := y(0), \quad \mathbb{E}^g[X|(B(s))_{s\in[0,t]}] := y(t).$$

- Artzner-Delbean-Eden-Heath1999, Coherent measures of risk, Math. finance.

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- State dependent Markovian case: Avellaneda, M., Levy, A. and Paras A. (1995), T. Lyons (1995).
- Longtime blockage...

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- [Peng2008-SPA] Multi-Dim G-Brownian Motion and Related Stochastic Calculus.
- [Denis-Hu-Peng2008] Capacity related to Sublinear Expectations: appl. to G-Brownian Motion Paths.

Related works

- [Peng2009] Survey on G-normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, Science in China Series A: Mathematics, Volume 52, Number 7, 1391-1411.

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- L. Epstein and S. Ji (2012) Ambiguous volatility, possibility and utility in continuous time, (by random G-expectations).

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- M. Soner, N. Touzi, and J. Zhang (2011) Martingale representation theorem for the G-expectation. in SPA.

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Related works

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- Large Deviations for Stochastic Differential Equations Driven by G-Brownian Motion. Stoch. Proc. Appl., 120 (2010)
- M. Hu, S. Ji, S. P. & S. Song, (2012) Backward Stochastic Differential Equations driven by G-Brownian Motions.

This talk is based on

- Peng, S. Notes: Nonlinear Expectations and Stochastic Calculus under Uncertainty, arxiv 2010.

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- Mingshang Hu, Shaolin Ji, Shige Peng, Yongsheng Song Backward Stochastic Differential Equations Driven by G-Brownian Motion, arXiv:1206.5889v1 [math.PR] (26 Jun.) 2012.

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- Ω: space of scenarios;

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- $\mathcal{H}$ a linear space of risk positions or (risk losses) containing constants (real functions defined on Ω) s.t.

$$X \in \mathcal{H} \implies |X| \in \mathcal{H}$$

Expectation framework–G-framework

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- $\mathcal{H}$ a linear space of risk positions or (risk losses) containing constants (real functions defined on Ω) s.t.

$$X \in \mathcal{H} \implies |X| \in \mathcal{H}$$

- We often "equivalently" assume:

$$X_1, \cdots, X_n \in \mathcal{H} \implies \varphi(X_1, \cdots, X_n) \in \mathcal{H}, \quad \forall \varphi \in C_{Lip}(\mathbb{R}^n)$$

$X \in \mathcal{H} \implies |X| \in \mathcal{H}$

(a) $E[X] \geq E[Y]$, if $X \geq Y$

$$X \in \mathcal{H} \implies |X| \in \mathcal{H}$$

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Daniell's Expectation: $(\Omega, \mathcal{H}, \mathbb{E})$ v.s. $(\Omega, \mathcal{F}, \mathbb{P})$

$X \in \mathcal{H} \implies |X| \in \mathcal{H}$

(a) $E[X] \geq E[Y]$, if $X \geq Y$

(b) $E[c] = c$,

(c) $E[X + Y] = E[X] + E[Y]$,

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- *For each given* $X \in \mathcal{H}$,

$$\hat{\mathbb{E}}[\varphi(X)] = \sup_{\theta \in \Theta} \int_{\mathbb{R}} \varphi(x) dF_\theta(x), \quad F_\theta(x) = P_\theta(X \leq x).$$

Robust representation of a coherent risk measure

- Huber Robust Statistics (1981), for finite state case.
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Meaning:

Sublinear expectation corresponds the Knightian uncertainty of probabilities: $\{P_\theta\}_{\theta\in\Theta}$

Uncertainty version of distributions in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

Definition

- $X \sim Y$ *if they have the same distribution uncertainty*

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- $$Y \text{ indenp. of } X \iff \hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Theorem

Let $\{X_i\}_{i=1}^{\infty}$ *in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *be i.i.d.:* $X_i \sim X_1$ *and* X_{i+1} *Indep.* $(X_1, \cdots, X_i)$*. Assume:*

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty \quad , \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

Then:

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}})] = \hat{\mathbb{E}}[\varphi(X)], \ \forall \varphi \in C_b(\mathbb{R}),$$

with $X \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$*, where*

$$\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2], \ \ \underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2].$$

Normal distributions under Knightian uncertainty

Definition

A loss position X in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is normally in uncertainty distribution if

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2} X, \quad \forall a, b \geq 0.$$

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Theorem.

If $(B_t(\omega))_{t\geq 0}$ is a G**–Brownian motion** and $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] \equiv 0$ then:
$B_{t+s} - B_s \overset{d}{=} N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, $\forall\ s, t \geq 0$ □

Construct G-BM on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

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- For each $X(\omega) = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$, with $t_i < t_{i+1}$, we set

$$\hat{\mathbb{E}}[X] := \tilde{\mathbb{E}}[\varphi(\sqrt{t_1}\xi_1, \sqrt{t_2 - t_1}\xi_2, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)]$$

where

$$\xi_i \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2]),\ \ \xi_{i+1} \text{ is indep. of } (\xi_1, \cdots, \xi_i) \text{ under } \tilde{\mathbb{E}}.$$

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- Conditional expectation:

$$\hat{\mathbb{E}}_{t_1}[X] = \tilde{\mathbb{E}}[\varphi(x, \sqrt{t_2 - t_1}\xi_2, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)]_{x=B_{t_1}}$$

Probability v.s. Nonlinear Expectation

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
Distributions: $X \overset{d}{=} Y$	$X \overset{d}{=} Y$,
Independence: Y indep. of X	Y indep. of X, (non-symm.)
LLN and CLT	LLN + CTL
Normal distributions	G-Normal distributions
Brownian motion $B_t(\omega) = \omega_t$	G-B.M. $B_t(\omega) = \omega_t$,
Qudratic variat. $\langle B \rangle_t = t$	$\langle B \rangle_t$: still a G-Brownian motion
Lévy process	G-Lévy process

Probability v.s. Nonlinear Expectation

Probability Space	**Nonlinear Expectation Space**
Itô's calculus for BM	Itô's calculus for G-BM
SDE $dx_t = b(x_t)dt + \sigma(x_t)dB_t$	$dx_t = \cdots + \beta(x_t)d\langle B\rangle_t$
Diffusion: $\partial_t u - \mathcal{L}u = 0$	$\partial_t u - G(Du, D^2u) = 0$
Markovian pro. and semi-grou	Nonlinear Markovian
Martingales	G-Martingales
$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$
	$K_t \overset{?}{=} \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$

Probability Space	**Nonlinear Expectation Space**
P-almost surely analysis	$\hat{c}$-quasi surely analysis
	$\hat{c}(A) = \sup_\theta E_{P_\theta}[\mathbf{1}_A]$
$X(\omega)$: P-quasi continuous	$X(\omega)$: $\hat{c}$-quasi surely
$\iff$ X is $\mathcal{B}(\Omega)$-meas.	continuous $\Longrightarrow$ X is $\mathcal{B}(\Omega)$-meas.

BSDE

Backward stochastic differential equations (BSDE) driven by a G-Brownian motion $(B_t)_{t\geq 0}$ in the following form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

Under a Lipschitz condition of f and g in Y and Z. The existence and uniqueness of the solution (Y, Z, K) is proved, where K is a decreasing G-martingale.

Representation of G-martingale

G-martingale M is of the form

$$M_t = M_0 + \bar{M}_t + K_t,$$
$$\bar{M}_t := \int_0^t z_s B_s,$$
$$K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds.$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s$$
$$- \int_t^T Z_s dB_s - (K_T - K_t).$$

Existing results on fully nonlinear BSDEs

- f independent of z (and $g = 0$):

$$Y_t^i = \hat{\mathbb{E}}_t^{G_i}[\xi^i + \int_t^T f^i(s, Y_s)ds].$$

Peng [2005,07,10].
BSDE corresponding to (path-depedent) system of PDE:

$$\begin{aligned} \partial_t u^i + G^i(u^i, Du^i, D^2u^i) + f^i(t, x, u^1, \cdots, u^k) &= 0, \\ u^i(x, T) &= \varphi^i(x), \\ i &= 1, \cdots, k. \end{aligned}$$

G^i satisfy the dominate condition:

$$G^i(x, y, p, A) - G^i(x, \bar{y}, \bar{p}, \bar{A}) \leq c(|y - \bar{y}| + |p - p|) + \hat{G}(A - \bar{A}),$$

Existing results on fully nonlinear BSDEs

- [Soner, Touzi and Zhang, 2BSDE]
- $(Y, Z, K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_H^{\kappa}}$, $\mathbb{P} \in \mathcal{P}_H^{\kappa}$, the following BSDE

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + (K_T^{\mathbb{P}} - K_t^{\mathbb{P}}), \quad \mathbb{P}\text{-a.s.},$$

with

-

$$K_t^{\mathbb{P}} = \operatorname{ess} \inf_{\mathbb{P}'\in\mathcal{P}_H^{\kappa}(t+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}}], \quad \mathbb{P}\text{-a.s.}, \quad \forall \mathbb{P} \in \mathcal{P}_H^{\kappa}, \ t \in [0, T].$$

A priori estimates

- $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$
- $\Omega_T = C_0([0,T], \mathbb{R})$,
- $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2] \geq -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2 > 0$.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \quad \text{(GBSDE)}$$

where

$$f(t, \omega, y, z) : [0,T] \times \Omega_T \times \mathbb{R}^2 \to \mathbb{R}$$

Assumption: some $\beta > 1$ such that

(H1) for any y, z, $f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$,

(H2) $|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|)$.

For sim(Y, Z, K) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$, K: a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Lemma 3.3.

Let X_t, X_t^n be as in the above Lemma and $\alpha^* = \frac{\alpha}{\alpha-1}$. Assume that K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^{\alpha^*}(\Omega_T)$. Then we have

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t X_s^n dK_s - \int_0^t X_s dK_s||] \to 0 \text{ as } n \to \infty.$$

□

An important observation

Lemma 3.4.

Let $X \in S_G^{\alpha}(0,T)$ for some $\alpha > 1$ and $\alpha^* = \frac{\alpha}{\alpha-1}$. Assume that K^j, $j = 1, 2$, are two decreasing G-martingales with $K_0^j = 0$ and $K_T^j \in L_G^{\alpha^*}(\Omega_T)$. Then the process defined by

$$\int_0^t X_s^+ dK_s^1 + \int_0^t X_s^- dK_s^2$$

is also a decreasing G-martingale. □

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$

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- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$
- $= |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2] - 2\int_t^T [(\hat{Y}_s)^- dK_t^1 + (\hat{Y}_s)^+ dK_t^2]$

A typical application of Lemma 3.4

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- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s \hat{f}_s ds - \int_t^T |\hat{Z}_t|^2 d\langle B \rangle_t + \int_t^T 2\hat{Y}_s \hat{Z}_s dB_s$
- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$
- $= |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2] - 2\int_t^T [(\hat{Y}_s)^- dK_t^1 + (\hat{Y}_s)^+ dK_t^2]$
- $\geq |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2]$

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$
- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s\hat{f}_s ds - \int_t^T |\hat{Z}_t|^2 d\langle B\rangle_t + \int_t^T 2\hat{Y}_s\hat{Z}_s dB_s$
- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$
- $= |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2] - 2\int_t^T [(\hat{Y}_s)^- dK_t^1 + (\hat{Y}_s)^+ dK_t^2]$
- $\geq |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2]$
- Thus

$$|\hat{Y}_t|^2 \leq \hat{\mathbb{E}}_t[|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s\hat{f}_s ds - \int_t^T |\hat{Z}_t|^2 d\langle B\rangle_t]$$

Proposition 3.5.

Assume (H1)-(H2) and $(Y, Z, K_T) \in \mathbb{S}^\alpha(0,T) \times \mathbb{H}^\alpha(0,T) \times \mathbb{S}^\alpha(\Omega_T)$ solves

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where K is a decreasing process with $K_0 = 0$. Then

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{1}{2}} (\hat{\mathbb{E}}[(\int_0^T |f_s^0| ds)^\alpha])^{\frac{1}{2}}\},$$

$$\hat{\mathbb{E}}[|K_T|^\alpha] \leq C_\alpha \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + \hat{\mathbb{E}}[(\int_0^T |f_s^0 ds)^\alpha]\},$$

$$f_s^0 := |f(s,0,0)| + L^w \varepsilon$$

Proposition 3.7.

We assume (H1) and (H2). Assume that $(Y, Z, K) \in \mathfrak{S}_G^{\alpha}(0, T)$ for some $1 < \alpha < \beta$ is a solution (GBSDE). Then

- There exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ such that

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f_s^0|^\alpha ds],$$

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \leq C_\alpha \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_0^T |f_s^0|^\alpha ds]],$$

  where $f_s^0 = |f(s, 0, 0)| + L^w \varepsilon$.

- For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, $\underline{\sigma}$, L^w such that

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \leq C_{\alpha,\alpha'}\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]]$$

$$+ (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}}$$

Proposition 3.8.

Let f_i, $i = 1, 2$, satisfy (H1) and (H2). Assume

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $Y^i \in \mathbb{S}^\alpha(0, T)$, $Z^i \in \mathbb{H}^\alpha(0, T)$, K^i is a decreasing process with $K_0^i = 0$ and $K_T^i \in \mathbb{L}^\alpha(\Omega_T)$ for some $\alpha > 1$. Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$. Then there exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ such that

$$\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{\|\hat{Y}\|_{\mathbb{S}^\alpha}^\alpha + \|\hat{Y}\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} \sum_{i=1}^2 [||Y^i||_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} + ||\int_0^T f_s^{i,0} ds||_{\alpha,G}^{\frac{\alpha}{2}}]\},$$

where $f_s^{i,0} = |f_i(s, 0, 0)| + L^w \varepsilon$, $i = 1, 2$. □

Proposition 3.9.

Let $\xi^i \in L_G^{\beta}(\Omega_T)$ with $\beta > 1$, $i = 1, 2$, and f_i satisfy (H1) and (H2). Assume that $(Y^i, Z^i, K^i) \in \mathfrak{S}_G^{\alpha}(0, T)$ for some $1 < \alpha < \beta$ are the solutions of equation (GBSDE) to ξ^i and f_i. Then

(i) $|\hat{Y}_t|^{\alpha} \le C_{\alpha} \hat{\mathbb{E}}_t[|\hat{\xi}|^{\alpha} + \int_t^T |\hat{f}_s|^{\alpha} ds]$, where $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)| + L_1^w \varepsilon$.

(ii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, $\underline{\sigma}$, L^w such that

$$\begin{aligned}\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\hat{Y}_t|^{\alpha}] \le C_{\alpha,\alpha'}\{&\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^{\alpha}]] \\ &+ (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} \\ &+ \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]]\}.\end{aligned}$$

□

Existence and uniqueness of G-BSDEs

$$\partial_t u + G(\partial^2_{xx} u) + h(u, \partial_x u) = 0, \quad u(T, x) = \varphi(x). \tag{GPDE}$$

We approximate the solution f by those of equations (GBSDE) with much simpler $\{f_n\}$. More precisely, assume that $\|f_n - f\|_{M_G^\beta} \to 0$ and (Y^n, Z^n, K^n) is the solution of the following G-BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n).$$

We try to prove that (Y^n, Z^n, K^n) converges to (Y, Z, K) and (Y, Z, K) is the solution of the following G-BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t).$$

Main results:

Theorem

Assume that $\xi \in L_G^{\beta}(\Omega_T)$, $\beta > 1$ *and* f *satisfies (H1) and (H2). Then equation (G-BSDE) has a unique solution* (Y, Z, K). *Moreover, for any* $1 < \alpha < \beta$ *we have* $Y \in S_G^{\alpha}(0, T)$, $Z \in H_G^{\alpha}(0, T)$ *and* $K_T \in L_G^{\alpha}(\Omega_T)$.

Sketch of Proof of Theorem.

Step 1. $f(t,\omega,y,z) = h(y,z)$, $h \in C_0^\infty(\mathbb{R}^2)$.
Part 1) $\xi = \varphi(B_T - B_{t_1})$: $\exists\ \alpha \in (0,1)$ s.t.,

$$||u||_{C^{1+\alpha/2,2+\alpha}([0,T-\kappa]\times\mathbb{R})} < \infty, \quad \kappa > 0.$$

Itô's formula to $u(t, B_t - B_{t_1})$ on $[t_1, T-\kappa]$, we get

$$\begin{aligned} u(t,B_t - B_{t_1}) =& u(T-\kappa, B_{T-\kappa} - B_{t_1}) + \int_t^{T-\kappa} h(u,\partial_x u)(s, B_s - B_{t_1})ds \\ & - \int_t^{T-\kappa} \partial_x u(s, B_s - B_{t_1})dB_s - (K_{T-\kappa} - K_t), \end{aligned}$$

□

Sketch of Proof of Theorem.

where

$$K_t = \frac{1}{2}\int_{t_1}^{t} \partial^2_{xx} u(\cdot)\,d\langle B\rangle_s - \int_{t_1}^{t} G(\partial^2_{xx} u(\cdot))\,ds$$

$$|u(t,x) - u(s,y)| \le L_1(\sqrt{|t-s|} + |x-y|).$$

$\tilde{u}$ is the solution of PDE:

$$\partial_t \tilde{u} + G(\partial^2_{xx}\tilde{u}) + h(\tilde{u}, \partial_x \tilde{u}) = 0,$$
$$\tilde{u}(T,x) = \varphi(x + x_0).$$

□

Sketch of Proof of Theorem.

$$u(t, x + x_0) \leq u(t, x) + L_\varphi |x_0| \exp(L_h(T - t)),$$

Since x_0 is arbitrary, we get $|u(t, x) - u(t, y)| \leq \hat{L}|x - y|$, where $\hat{L} = L_\varphi \exp(L_h T)$. From this we can get $|\partial_x u(t, x)| \leq \hat{L}$ for each $t \in [0, T]$, $x \in \mathbb{R}$. On the other hand, for each fixed $\bar{t} < \hat{t} < T$ and $x \in \mathbb{R}$, applying Itô's formula to $u(s, x + B_s - B_{\bar{t}})$ on $[\bar{t}, \hat{t}]$, we get

$$u(\bar{t}, x) = \hat{\mathbb{E}}[u(\hat{t}, x + B_{\hat{t}} - B_{\bar{t}}) + \int_{\bar{t}}^{\hat{t}} h(u, \partial_x u)(s, x + B_s - B_{\bar{t}}) ds].$$

□

Sketch of Proof of Theorem.

From this we deduce that

$$|u(\bar{t},x)-u(\hat{t},x)| \leq \hat{\mathbb{E}}[\hat{L}|B_{\hat{t}}-B_{\bar{t}}|+\tilde{L}|\hat{t}-\bar{t}|] \leq (\hat{L}\bar{\sigma}+\tilde{L}\sqrt{T})\sqrt{|\hat{t}-\bar{t}|},$$

where $\tilde{L}=\sup_{(x,y)\in\mathbb{R}^2}|h(x,y)|$. Thus we get (**??**) by taking $L_1=\max\{\hat{L},\hat{L}\bar{\sigma}+\tilde{L}\sqrt{T}\}$. Letting $\kappa\downarrow 0$ in Itô's equation, it is easy to verify that

$$\hat{\mathbb{E}}[|Y_{T-\kappa}-\xi|^2+\int_{T-\kappa}^{T}|Z_t|^2dt+(K_{T-\kappa}-K_T)^2]\to 0,$$

where $Y_t=u(t,B_t-B_{t_1})$ and $Z_t=\partial_x u(t,B_t-B_{t_1})$. Thus $(Y_t,Z_t,K_t)_{t\in[t_1,T]}$ is a solution of equation (GBSDE) with terminal value $\xi=\varphi(B_T-B_{t_1})$. Furthermore, it is easy to check that $Y\in S_G^{\alpha}(t_1,T)$, $Z\in H_G^{\alpha}(t_1,T)$ and $K_T\in L_G^{\alpha}(\Omega_T)$ for any $\alpha>1$. □

Sketch of Proof of Theorem.

Part 2) $\xi = \psi(B_{t_1}, B_T - B_{t_1})$:

$$u(t,x,B_t - B_{t_1}) = u(T,x,B_T - B_{t_1}) + \int_t^T h(u, \partial_y u)(s,x,B_s - B_{t_1})ds$$
$$- \int_t^T \partial_y u(\cdot) dB_s - (K_T^x - K_t^x),$$

$$K_t^x = \frac{1}{2}\int_{t_1}^t \partial_{yy}^2 u(\cdot) d\langle B\rangle_s - \int_{t_1}^t G(\partial_{yy}^2 u(\cdot))ds.$$

$$Y_t = Y_T + \int_t^T h(Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

□

Sketch of Proof of Theorem.

where

$$Y_t := u(t, B_{t_1}, B_t - B_{t_1}), \quad Z_t := \partial_y u(\cdot),$$
$$K_t := \frac{1}{2}\int_{t_1}^t \partial^2_{yy} u(\cdot) d\langle B\rangle_s - \int_{t_1}^t G(\partial^2_{yy} u(\cdot)) ds.$$

Need to prove $(Y, Z, K) \in \mathfrak{S}^\alpha_G(0, T)$. By partition of unity theorem, $\exists$ $h_i^n \in C_0^\infty(\mathbb{R})$ s.t.

$$\lambda(\text{supp}(h_i^n)) < 1/n, \quad 0 \leq h_i^n \leq 1,$$
$$I_{[-n,n]}(x) \leq \sum_{i=1}^{k_n} h_i^n \leq 1.$$

$\square$

Sketch of Proof of Theorem.

We have

$$Y_t^n = Y_T^n + \int_t^T \sum_{i=1}^n h(y_s^{n,i}, z_s^{n,i}) h_i^n(B_{t_1}) ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n),$$

where

$$y_t^{n,i} = u(t, x_i^n, B_t - B_{t_1}), \quad z_t^{n,i} = \partial_y u(t, x_i^n, B_t - B_{t_1}),$$

$$Y_t^n = \sum_{i=1}^n y_t^{n,i} h_i^n(B_{t_1}), \quad Z_t^n = \sum_{i=1}^n z_t^{n,i} h_i^n(B_{t_1}),$$

$$K_t^n = \sum_{i=1}^n K_t^{x_i^n} h_i^n(B_{t_1}).$$

□

Sketch of Proof of Theorem.

Thus

$$|Y_t - Y_t^n| \leq \sum_{i=1}^{k_n} h_i^n(B_{t_1})|u(t, x_i^n, B_t - B_{t_1}) - u(t, B_{t_1}, B_t - B_{t_1})|$$

$$+ |Y_t| I_{[|B_{t_1}|>n]} \leq \frac{L_2}{n} + \frac{||u||_\infty}{n}|B_{t_1}|.$$

Thus

$$\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^\alpha] \leq \hat{\mathbb{E}}[(\frac{L_2}{n} + \frac{||u||_\infty}{n}|B_{t_1}|)^\alpha] \to 0.$$

By the estimates

$$\hat{\mathbb{E}}[(\int_{t_1}^T |Z_s - Z_s^n|^2 ds)^{\alpha/2}] \leq C_\alpha\{\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^\alpha]$$

$$+(\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^\alpha])^{1/2}\} \to 0.$$

Thus $Z \in M_G^\alpha(0,T)$, $K_t \in L_G^\alpha(\Omega_t)$.

Sketch of Proof of Theorem.

[Sketch of Proof of Theorem] prove K is G-martingale. Following [Li-P.], we take

$$h_i^n(x) = I_{[-n+\frac{i}{n},-n+\frac{i+1}{n})}(x), \quad i=0,\ldots, \quad 2n^2-1,$$
$$h_{2n^2}^n = 1 - \sum_{i=0}^{2n^2-1} h_i^n$$

$$\tilde{Y}_t^n = \sum_{i=0}^{2n^2} u(t,-n+\frac{i}{n}, B_t - B_{t_1})h_i^n(B_{t_1}), \ \tilde{Z}_t^n = \sum_{i=0}^{2n^2} \partial_y u(\cdot)h_i^n(B_{t_1})$$

solves

$$\tilde{Y}_t^n = \tilde{Y}_T^n + \int_t^T h(\tilde{Y}_s^n, \tilde{Z}_s^n)ds - \int_t^T \tilde{Z}_s^n dB_s - (\tilde{K}_T^n - \tilde{K}_t^n),$$

□

Sketch of Proof of Theorem.

We have $\hat{\mathbb{E}}[(\int_{t_1}^{T}|Z_s-\tilde{Z}_s^n|^2ds)^{\alpha/2}]\rightarrow 0$. Thus $\hat{\mathbb{E}}[|K_t-\tilde{K}_t^n|^{\alpha}]\rightarrow 0$ and $\hat{\mathbb{E}}_t[K_s]=K_t$. For $Y_{t_1}=u(t_1,B_{t_1},0)$, we can use the same method as Part 1 on $[0,t_1]$.

Step 2) $f(t,\omega,y,z)=\sum_{i=1}^{N}f^ih^i(y,z)$ with $f^i\in M_G^0(0,T)$ and $h^i\in C_0^{\infty}(\mathbb{R}^2)$.

□

Sketch of Proof of Theorem.

Step 3) $f(t,\omega,y,z)=\sum_{i=1}^N f^i h^i(y,z)$ with $f^i\in M_G^\beta(0,T)$ bounded and $h^i\in C_0^\infty(\mathbb{R}^2)$, $h^i\geq 0$ and $\sum_{i=1}^N h^i\leq 1$:
Choose

$$f_n^i\in M_G^0(0,T) \text{ s.t. } |f_n^i|\leq \|f^i\|_\infty,\ \sum_{i=1}^N \|f_n^i-f^i\|_{M_G^\beta}<1/n.$$

Set $f_n:=\sum_{i=1}^N f_n^i h^i(y,z)$.
Let (Y^n,Z^n,K^n) be the solution of (GBSDE) with generator f_n.

$$\begin{aligned}\hat{f}_s^{m,n} &:= |f_m(s,Y_s^n,Z_s^n)-f_n(s,Y_s^n,Z_s^n)|\\ &\leq \sum_{i=1}^N |f_n^i-f^i|+\sum_{i=1}^N |f_m^i-f^i| =: \hat{f}_n+\hat{f}_m,\end{aligned}$$

□

Sketch of Proof of Theorem.

We have, for any $1 < \alpha < \beta$,

$$\hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s^{m,n} ds)^\alpha] \leq \hat{\mathbb{E}}_t[(\int_0^T (|\hat{f}_n(s)| + |\hat{f}_m(s)|) ds)^\alpha].$$

By Theorem 2.10, $\forall \alpha \in (1, \beta)$

$$\hat{\mathbb{E}}\left[\sup_t \hat{\mathbb{E}}_t[|\int_0^T \hat{f}_s^{m,n} ds|^\alpha]]\right] \to 0, \ \ m, n \to \infty$$

By Proposition 3.9 $\{Y^n\}$ is Cauchy under $\|\cdot\|_{S_G^\alpha}$. By Proposition 3.7, 3.8, $\{Z^n\}$ is a also Cauchy under $\|\cdot\|_{H_G^\alpha}$ thus $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$ under $\|\cdot\|_{L_G^\alpha}$ thus $\{K_T^n\}$ is also Cauchy under $\|\cdot\|_{L_G^\alpha}$.

□

Sketch of Proof of Theorem.

Step 4). f is bounded, Lipschitz. $|f(t,\omega,y,z)| \leq CI_{B(R)}(y,z)$ for some $C, R > 0$. Here $B(R) = \{(y,z)|y^2+z^2 \leq R^2\}$.
For any n, by the partition of unity theorem, there exists $\{h_n^i\}_{i=1}^{N_n}$ such that $h_n^i \in C_0^\infty(\mathbb{R}^2)$, the diameter of support $\lambda(\text{supp}(h_n^i)) < 1/n$, $0 \leq h_n^i \leq 1$, $I_{B(R)} \leq \sum_{i=1}^N h_n^i \leq 1$. Then $f(t,\omega,y,z) = \sum_{i=1}^N f(t,\omega,y,z)h_n^i$. Choose y_n^i, z_n^i such that $h_n^i(y_n^i, z_n^i) > 0$. Set

$$f_n(t,\omega,y,z) = \sum_{i=1}^N f(t,\omega,y_n^i,z_n^i)h_n^i(y,z)$$

□

Sketch of Proof of Theorem.

Then

$$|f(t,\omega,y,z)-f_n(t,\omega,y,z)| \leq \sum_{i=1}^{N} |f(t,\omega,y,z)-f(t,\omega,y_n^i,z_n^i)|h_n^i \leq L/n$$

and

$$|f_n(t,\omega,y,z)-f_n(t,\omega,y',z')| \leq L(|y-y'|+|z-z'|+2/n).$$

Noting that $|f_m(s,Y_s^n,Z_s^n)-f_n(s,Y_s^n,Z_s^n)| \leq (L/n+L/m)$, □

Sketch of Proof of Theorem.

we have

$$\hat{\mathbb{E}}_t[|\int_0^T (|f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| + \frac{2L}{m})ds|^\alpha] \leq T^\alpha(\frac{L}{n} + \frac{3L}{m})^\alpha.$$

So by the estimates $\{Y^n\}$ cauchy under $\|\cdot\|_{S_G^\alpha}$. $\{Z^n\}$ is cauchy under $\|\cdot\|_{H_G^\alpha}$. is also cauchy $\{\int_0^T f_n(s, Y_s^n, Z_s^n)ds\}$ under $\|\cdot\|_{L_G^\alpha}$. □

Sketch of Proof of Theorem.

Step 5). f is bounded, Lipschitz.
For any $n \in \mathbb{N}$, choose $h^n \in C_0^\infty(\mathbb{R}^2)$ such that $I_{B(n)} \le h^n \le I_{B(n+1)}$ and $\{h^n\}$ are uniformly Lipschitz w.r.t. n. Set $f_n = fh^n$, which are uniformly Lipschitz. Noting that for $m > n$

$$
\begin{aligned}
&|f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| \\
&\le |f(s, Y_s^n, Z_s^n)| I_{[|Y_s^n|^2 + |Z_s^n|^2 > n^2]} \\
&\le \|f\|_\infty \frac{|Y_s^n| + |Z_s^n|}{n},
\end{aligned}
$$

□

Sketch of Proof of Theorem.

we have

$$\begin{aligned}
&\hat{\mathbb{E}}_t[(\int_0^T |f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)|ds)^\alpha] \\
&\leq \frac{\|f\|_\infty^\alpha}{n^\alpha}\hat{\mathbb{E}}_t[(\int_0^T |Y_s^n| + |Z_s^n|ds)^\alpha] \\
&\leq \frac{\|f\|_\infty^\alpha}{n^\alpha}C(\alpha, T)\hat{\mathbb{E}}_t[\int_0^T |Y_s^n|^\alpha ds + (\int_0^T |Z_s^n|^2 ds)^{\alpha/2}],
\end{aligned}$$

where $C(\alpha, T) := 2^{\alpha-1}(T^{\alpha-1} + T^{\alpha/2}])$. □

Sketch of Proof of Theorem.

So by Theorem 2.10 and Proposition 3.4 we get $|| \int_0^T \hat{f}_s^{m,n} ds||_{\alpha,\mathcal{E}} \longrightarrow 0$ as $m, n \longrightarrow \infty$ for any $\alpha \in (1, \beta)$. By Proposition 3.5, we conclude that $\{Y^n\}$ is cauchy under $\|\cdot\|_{S_G^\alpha}$. $\{Z^n\}$ cauchy sequence under $\|\cdot\|_{H_G^\alpha}$. $\{\int_0^T f_n(s, Y_s^n, Z_s^n)ds\}$ is cauchy under $\|\cdot\|_{L_G^\alpha}$:

$$\begin{aligned} &|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\ &\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\ &\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + |f(s, Y_s^n, Z_s^n)| 1_{[|Y_s^n|+|Z_s^n|>n]}, \end{aligned}$$

which implies the desired result. □

Sketch of Proof of Theorem.

Step 6). For the general f.
Set $f_n = [f \vee (-n)] \wedge n$, which are uniformly Lipschitz. Choose $0 < \delta < \frac{\beta-\alpha}{\alpha} \wedge 1$. Then $\alpha < \alpha' = (1+\delta)\alpha < \beta$. Since for $m > n$

$$|f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \leq |f(s, Y^n_s, Z^n_s)| I_{[|f(s, Y^n_s, Y^n_s)|>n]} \leq \frac{1}{n^\delta}|f(s, Y^n_s$$

we have

$$\begin{aligned}
&\hat{\mathbb{E}}_t[(\int_0^T |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)|ds)^\alpha] \\
&\leq \frac{1}{n^{\alpha\delta}} \hat{\mathbb{E}}_t[(\int_0^T |f(s, Y^n_s, Z^n_s)|^{1+\delta} ds)^\alpha], \\
&\leq \frac{C(\alpha, T, L, \delta)}{n^{\alpha\delta}} \hat{\mathbb{E}}_t[\int_0^T |f(s, 0, 0)|^{\alpha'} ds + \int_0^T |Y^n_s|^{\alpha'} ds + (\int_0^T |Z^n_s|^2 ds)^{\frac{\alpha'}{2}}],
\end{aligned}$$

where $C(\alpha, T, L, \delta) := 3^{\alpha'-1}(T^{\alpha-1} + L^{\alpha'} T^{\frac{\alpha(1-\delta)}{2}} + T^{\alpha-1} L^{\alpha'})$. □

Sketch of Proof of Theorem.

So by Song's estimate and a priori estimate, we get $||\int_0^T \hat{f}_s^{m,n} ds||_{\alpha,\mathcal{E}} \longrightarrow 0$ as $m, n \longrightarrow \infty$ for any $\alpha \in (1, \beta)$. We know that $\{Y^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. And consequently $\{Z^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{H_G^\alpha}$. Now we prove $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$. In fact,

$$\begin{aligned}
&|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\
&\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + \frac{3^\delta}{n^\delta}(|f_s^0|^{1+\delta} + |Y_s^n|^{1+\delta} + |Z_s^n|^{1+\delta}),
\end{aligned}$$

which implies the desired result. □

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