Uniform logarithmic Sobolev inequalities in dimension for harmonic measures on spheres

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Outline

1. Introduction
2. Uniform Poincaré inequality
3. Uniform logarithmic Sobolev inequality

Harmonic measures

Let S^{n-1} be the unit sphere on $\mathbb{R}^n$ ($n \geq 3$) and μ the normalized Lebesgue measure on S^{n-1}, i.e. $\mu = \sigma_{n-1}/s_{n-1}$, where σ_{n-1} and s_{n-1} are the uniform surface measure and total volume respectively on S^{n-1}.

For $x = (x_1, \cdots, x_n)$ with $|x| < 1$, the harmonic measure μ_x^n is one probability on S^{n-1} given by

$$d\mu_x^n(y) = \frac{1 - |x|^2}{|y - x|^n} d\mu(y), \quad y \in S^{n-1}.$$

Harmonic measures

If f is an integrable function on S^{n-1},

$$\tilde{f}(x) := \int_{S^{n-1}} f(y) d\mu_x^n(y)$$

is a harmonic function whose radical limits are equal μ-almost everywhere to f.

Moreover, if $\mathbb{P}^x$ denotes the probability distribution of a standard n-dimensional Brownian motion B_t starting from x, and τ the first time for B_t to hit S^{n-1}, μ_x^n is nothing but the distribution of B_τ under $\mathbb{P}^x$. 1944, Kakutani.

Wasserstein distance

Let M be a connected complete Riemannian manifold with Riemannian metric ρ and ∇ is the gradient on M. $\mathcal{M}_1(M)$ is the space of all probabilities on M.

For any $\nu, \mu \in \mathcal{M}_1(M)$, the L^p-Wasserstein distance $W_{p,\rho}(\nu, \mu)$ associated with the metric ρ is defined as

$$W_{p,\rho}(\nu, \mu) = \inf_{\pi} \left(\int \int_{M^2} \rho^p(x, y)\pi(dx, dy) \right)^{1/p}$$

where π runs over all couplings of (ν, μ).

Entropy and Fisher information

- The relative entropy of ν w.r.t. μ is given by

$$H(\nu/\mu) = \begin{cases} \int_M \log \frac{d\nu}{d\mu} d\nu, & \text{if } \nu << \mu, \\ +\infty, & \text{otherwise.} \end{cases}$$

- The Fisher-Donsker-Varadhan information of a probability measure ν w.r.t. μ is defined by

$$I(\nu/\mu) = \begin{cases} \int_M |\nabla\sqrt{f}|^2 d\mu, & \text{if } \nu = f\mu, \quad \text{and } \int_M |\nabla\sqrt{f}|^2 d\mu < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Related inequalities

Given any $\mu \in \mathcal{M}_1(M)$ we say that

- μ satisfies Poincaré inequality if there exists positive constant C such that for any smooth function $f : M \to \mathbb{R}$,

$$\mathrm{Var}_\mu(f) = \int_M f^2 d\mu - (\int_M f d\mu)^2 \le C \int_M |\nabla f|^2 d\mu.$$

  C_P is the optimal constant and the inequality is denoted as $\mu \in PI(C_{\mathrm{P}})$.

- μ satisfies logarithmic Sobolev inequality with positive constant C ($\mu \in HI(C)$) if for any nonnegative smooth function $f : M \to \mathbb{R}_+$ with $\mu(f) = 1$,

$$H(f\mu/\mu) \le 2CI(\sqrt{f}, \sqrt{f}).$$

  The optimal constant is denoted by C_{LS}.

Transportation (information) inequality

- μ satisfies L^p-transportation inequality with positive constant C ($\mu \in W_pH(C)$) if

$$W_{p,\rho}^2(\nu,\mu) \le 2CH(\nu/\mu), \quad \forall \nu \in \mathcal{M}_1(M).$$

- μ satisfies L^p transportation-information inequality with positive constant C ($\mu \in W_pI(C)$) if for any nonnegative smooth function $f : M \to \mathbb{R}_+$ with $\mu(f) = 1$

$$W_{p,\rho}^2(f\mu,\mu) \le 4C^2 I(\sqrt{f},\sqrt{f}).$$

Relations among the inequalities

- $HI(C) \Rightarrow W_2H(C)$,
  1. Otto-Villani, 00, JFA.
  2. Bobkov-Gentil-Ledoux 01, J. Math. Pures Appl.
  3. Cattiaux-Gullin, 06, J. Math. Pures Appl.
- $HI(C) \Rightarrow W_2I(C) \Rightarrow W_2H(C)$
  Guillin-Joulin-Wang-Wu, 09+
- $W_2H(C) \Rightarrow PI(C)$,
  Otto-Villani, 00, JFA

Known results on harmonic measures

- G. Schechtman and M. Schmuckenschläger proved that μ_x^n with any $|x| < 1$ have a uniform Gaussian concentration, 1995.
- F. Barthe and Z. Zhang proved that for $n \geq 3$, $\mu_x^n, |x| < 1$ satisfies uniform Poincaré, Talagrand inequalities and locally uniform Log-S inequalities for μ_x^n with $|x| \leq a < 1$, while logarithmic Sobolev constants explode with speed $\log \frac{1}{1-a}$ as $a \to 1$. 06+
- Zhang, harmonic measures on the unit circle. Uniform log-S inequality fails with same exploding speed $\log \frac{1}{1-a}$ as $a \to 1$. However uniform Poincaré, $W_1 I$ and $W_2 H$ inequlities hold with

$$C_{\mathrm{P}} \leq 4, \; C_{W_1 I} \leq \frac{2e}{e+1}, \; C_{W_2 H} \leq 25.3.$$

Fundamental Lemma 1: Barthe-Zhang

Lemma 1. Let M be a probability measure on S^{n-1}, such that $M(\{e_1, -e_1\}) = 0$, and ν the image of M by the mapping $y \to d(y, e_1)$. Assume that $M \in PI(C_{\mathrm{P}})$ or $M \in HI(C_{\mathrm{LS}})$ respectively, then $\nu \in PI(C_{\mathrm{P}})$ or $\nu \in HI(C_{\mathrm{LS}})$ respectively.

Fundamental Lemma 2: Barthe-Zhang

Lemma 2. Let M be a probability measure on S^{n-1} with

$$dM(x) = \varphi(d(x, e_1))d\sigma_{n-1}(x), \quad x \in S^{n-1},$$

where φ is measurable. Let ν be the image of M by the mapping $x \to d(x, e_1)$. Assume that $\nu \in PI(C_{\mathrm{P}})$ or $\nu \in HI(C_{\mathrm{LS}})$ respectively, then $M \in PI(\max\{C_{\mathrm{P}}, c_{n-2}\}$ or $M \in HI(\max\{C_{\mathrm{LS}}, c_{n-2}\})$ respectively, where $c_{n-2} = \frac{1}{n-2}$ is the optimal Poincaré and logarithmic Sobolev constant of the uniform probability σ_{n-2}/s_{n-2} on S^{n-2}.

Reducation to one dimensional case: Barthe-Zhang

Take $x = ae_1$ with $0 \leq a < 1$. Let $\nu_{a,n}$ be the image of $\mu^n_{ae_1}$ by the mapping $y \to d(y, e_1)$. It is a probability on $[0, \pi]$ with density

$$\rho_{a,n} := \frac{d\nu_{a,n}}{d\theta} = (1 - a^2)\frac{s_{n-2}}{s_{n-1}} \frac{(\sin\theta)^{n-2}}{(1 + a^2 - 2a\cos\theta)^{n/2}}.$$

Uniform Poincaré inequality

Theorem. For any $x \in \mathbb{R}^n$ with $|x| < 1$ fixed, there exists positive constants $C_{\mathrm{P}}(n)$ independent of x such that for any smooth function $f : S^{n-1} \to \mathbb{R}$,

$$\mathrm{Var}_{\mu_x^n}(f) \leq C_{\mathrm{P}}(n) \int_{S^{n-1}} |\nabla_{S^{n-1}} f|^2 d\mu_x^n.$$

Moreover, $C_{\mathrm{P}}(n) \sim 1/n$.

Characterization available for C_{P}

- B. Muckenhoupt. Hardy inequalities with weights, *Studia Math.*, 1972.
- M-F Chen. Speed of stability for birth death processes, *Front. Math. China,* 2010.

Muchenhoupt's characterization

Let μ, ν be Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and $d\nu(x) = \rho(x)dx$, m be a median of μ. Assume that C_P is the optimal constant such that for every smooth function $f : \mathbb{R} \to \mathbb{R}$, one has

$$\mathrm{Var}_\mu(f) \le C_P \int f'^2 d\nu.$$

Then $\min\{b_{\mathrm{P}}, B_{\mathrm{P}}\} \le C_P \le 4 \max\{b_{\mathrm{P}}, B_{\mathrm{P}}\}$, where

$$b_{\mathrm{P}} = \sup_{x<m} \mu((-\infty, x]) \int_x^m \frac{dt}{\rho(t)}, \quad B_{\mathrm{P}} = \sup_{x>m} \mu([x, \infty)) \int_m^x \frac{dt}{\rho(t)}.$$

Chen's new characterization

Let $a > 0$, a and b be continuous on $[-M, N]$ (or $(-M, N]$ if $M = \infty$, for instance). Assume that $\mu(-M, N) < \infty$. Then for C_{P}, we have $\kappa \le C_{\mathrm{P}} \le 4\kappa$, where

$$\kappa = sup_{-M<y<x<N}\left[\left(\int_{-M}^{x} d\mu\right)^{-1} + \left(\int_{y}^{N} d\mu\right)^{-1}\right]^{-1} \int_{x}^{y} d\nu,$$

where

$$d\mu(x) = \frac{e^{C(x)}}{a(x)}dx, \quad d\nu(x) = e^{-C(x)}dx \quad \text{with} \quad C(x) = \int_{x_0}^{x} \frac{b}{a}.$$

Uniform Poincaré inequality

$$S_P^-(n) := \sup_{\alpha \in (0, m_{a,n})} \left(\int_0^\alpha \rho_{a,n}(\theta) d\theta \right) \left(\int_\alpha^{m_{a,n}} \frac{1}{\rho_{a,n}(\theta)} d\theta \right)$$

$$= \sup_{\alpha \in (0, m_{a,n})} \int_0^\alpha \frac{(\sin\theta)^{n-2}}{(1 + a^2 - 2a\cos\theta)^{n/2}} d\theta \int_\alpha^{m_{a,n}} \frac{(1 + a^2 - 2a\cos\theta)^{n/2}}{(\sin\theta)^{n-2}} d\theta$$

$$\sim \sup_{u \in (u_{a,n}, 1)} \int_u^1 \left(\frac{1 - t^2}{1 + a^2 - 2at} \right)^{\frac{n-3}{2}} dt \int_{u_{a,n}}^u \left(\frac{1 + a^2 - 2at}{1 - t^2} \right)^{\frac{n-1}{2}} dt,$$

where $m_{a,n}$ is the median and $u_{a,n} := cos m_{a,n}$.

Uniform Poincaré inequality—continued

$$S_{\mathrm{P}}^{+}(n) := \sup_{\alpha\in(m_{a,n},\pi)} \left(\int_{\alpha}^{\pi} \rho_{a,n}(\theta)d\theta\right)\left(\int_{m_{a,n}}^{\alpha} \frac{1}{\rho_{a,n}(\theta)}d\theta\right)$$

$$= \sup_{\alpha\in(m_{a,n},\pi)} \int_{\alpha}^{\pi} \frac{(\sin\theta)^{n-2}}{(1+a^2-2a\cos\theta)^{n/2}}d\theta \int_{m_{a,n}}^{\alpha} \frac{(1+a^2-2a\cos\theta)^{n/2}}{(\sin\theta)^{n-2}}d\theta$$

$$\sim \sup_{u\in(-1,u_{a,n})} \int_{-1}^{u} \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \int_{u}^{u_{a,n}} \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt.$$

On the median $m_{a,n}$

Lemma. Recall $u_{a,n} = cosm_{a,n}$. Then

$$u_{a,n} \geq a \quad \text{and } u_{a,n} - a \leq O(1/\sqrt{n}) \quad \text{as } n \text{ is large enough.}$$

Sketch of the proof. Let $\theta_a = \arccos a$. Aim: $m_{a,n} \geq \theta_a$.
Make a change of variables

$$\Phi(\theta) = \arccos \frac{\cos\theta - a}{\sqrt{1 + a^2 - 2a\cos\theta}},$$

which is increasing and $\Phi([0, \theta_a]) = [0, \pi/2]$, $\Phi([\theta_a, \pi]) = [\pi/2, \pi]$.

$$\begin{aligned}\int_0^{\alpha} \rho_{a,n}(\theta)d\theta &= \int_0^{\alpha} (1 - a^2)\frac{s_{n-2}}{s_{n-1}} \frac{(\sin\theta)^{n-2}}{(1 + a^2 - 2a\cos\theta)^{n/2}} d\theta \\ &= \int_0^{\Phi(\alpha)} (1 - a^2) G_a(\cos\phi)\rho_{0,n}(\phi)d\phi,\end{aligned}$$

where $G_a(c)$ a increasing function in c on $[-1, 1]$ and $G_a(0) = \frac{1}{1-a^2}$.

Key estimates

$$\forall u \in (u_{a,n}, 1), \quad \begin{cases} \int_u^1 \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \le O(\frac{1}{\sqrt{n}}), \\ \int_{u_{a,n}}^u \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt \le O(\frac{1}{\sqrt{n}}). \end{cases}$$

The two different integrals above achieve the order $1/\sqrt{n}$ as n large enough only if $u = a + O(1/\sqrt{n})$.

1. $0 < u - a = o(1/\sqrt{n})$;
2. $0 < u - a = O(1/\sqrt{n})$;
3. $1/\sqrt{n} = o(u-a), \ u \ge a$.

Key estimates—continued

$$\forall u \in (-1, u_{a,n}), \quad \begin{cases} \int_{-1}^{u} \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \le O(\frac{1}{\sqrt{n}}), \\ \int_{u}^{u_{a,n}} \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt \le O(\frac{1}{\sqrt{n}}). \end{cases}$$

The orders $1/\sqrt{n}$ can be achieved only if $|u-a| \le O(1/\sqrt{n})$.

1. $|u-a| = o(1/\sqrt{n})$;
2. $|u-a| = O(1/\sqrt{n})$;
3. $1/\sqrt{n} = o(|u-a|)$.

$$S_P^-(n) \sim \sup_{u\in(u_{a,n},1)} \int_u^1 \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \int_{u_{a,n}}^u \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt$$

$$S_P^+(n) \sim \sup_{u\in(-1,u_{a,n})} \int_{-1}^u \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \int_u^{u_{a,n}} \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt$$

Uniform logarithmic Sobolev inequality

Main results

Theorem. For any $x \in \mathbb{R}^n$ with $|x| < 1$ fixed, there exists one parameter $C_{\mathrm{LS}}(n)$ independent of x such that

$$\mathrm{Ent}_{\mu_x^n}(f^2) \leq C_{\mathrm{LS}}(n) \int_{S^{n-1}} |\nabla_{S^{n-1}} f|^2 d\mu_x^n.$$

Moreover $C_{\mathrm{LS}} \sim 1/n$ as n large enough.

Characterization available for C_{LS}

- S. G. Bobkov and F. Götze, Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.* , 1999.
- F. Barthe and C. Roberto, Sobolev inequalities for probabilty measures on the real line, *Studia Math.,* 2003.

Barthe-Roberto's Characterization for Log-S inequality

Let μ, ν be Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and $d\nu(x) = n(x)dx$, m be a median of μ. Suppose that C_{LS} is the optimal constant such that for every smooth function $f : \mathbb{R} \to \mathbb{R}$, one has

$$\mathrm{Ent}_\mu(f^2) \le C_{\mathrm{LS}} \int f'^2 d\nu.$$

Then $\max(b_{\mathrm{LS}}^-, b_{\mathrm{LS}}^+) \le C_{\mathrm{LS}} \le 4\max(B_{\mathrm{LS}}^-, B_{\mathrm{LS}}^+)$, where

$$b_{\mathrm{LS}}^+ = \sup_{x>m} \mu([x,\infty)) \log\left(1 + \frac{1}{2\mu([x,\infty))}\right) \int_m^x \frac{1}{n},$$

$$B_{\mathrm{LS}}^+ = \sup_{x>m} \mu([x,\infty)) \log\left(1 + \frac{e^2}{\mu([x,\infty))}\right) \int_m^x \frac{1}{n},$$

$$b_{\mathrm{LS}}^- = \sup_{x<m} \mu((-\infty,x]) \log\left(1 + \frac{1}{2\mu((-\infty,x])}\right) \int_x^m \frac{1}{n},$$

$$B_{\mathrm{LS}}^- = \sup_{x<m} \mu((-\infty,x]) \log\left(1 + \frac{e^2}{\mu((-\infty,x])}\right) \int_m^x \frac{1}{n}.$$

Uniform Log-S inequality

$$S_{LS}^{-}(n) := \sup_{\alpha\in(0,m_{a,n})} \left(\int_0^{\alpha} \rho_{a,n}\right) \log\left(1+\frac{1}{\int_0^{\alpha}\rho_{a,n}}\right)\left(\int_{\alpha}^{m_{a,n}}\frac{1}{\rho_{a,n}}\right)$$

$$\sim \sup_{u\in(u_{a,n},1)} \int_u^1 \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \log\left(1+\frac{1/\sqrt{n}}{\int_u^1 \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt}\right)$$

$$\cdot \int_{u_{a,n}}^{u} \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt.$$

Uniform Log-S inequality

$$S^+_{LS}(n) := \sup_{\alpha\in(m_{a,n},\pi)} \left(\int_\alpha^\pi \rho_{a,n}\right) \log\left(1+\frac{1}{\int_\alpha^\pi \rho_{a,n}}\right)\left(\int_{m_{a,n}}^\alpha \frac{1}{\rho_{a,n}}\right)$$

$$\sim \sup_{u\in(-1,u_{a,n})} \int_{-1}^u \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt \log\left(1+\frac{1/\sqrt{n}}{\int_{-1}^u \left(\frac{1-t^2}{1+a^2-2at}\right)^{\frac{n-3}{2}} dt}\right)$$

$$\cdot \int_u^{u_{a,n}} \left(\frac{1+a^2-2at}{1-t^2}\right)^{\frac{n-1}{2}} dt.$$

Corollary. For any $x \in \mathbb{R}^n$ with $|x| < 1$ fixed, the harmonic measure μ_x^n satisfies uniform $W_2H(C_{W_pH}(n))$ and $W_2I(C_{W_pI}(n))$ in dimension. Moreover
$$C_{W_2H}(n) \sim C_{W_2I}(n) \sim \frac{1}{n}.$$

Thanks for your attention