Generalized stochastic flow associated to Itô's SDE with partially Sobolev coefficients and its application

Dejun Luo

Institute of Applied Mathematics, AMSS, CAS

School of Mathematical Sciences, Beijing Normal University

2012-7-17

Outline

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Cauchy–Lipschitz theory

It is well known that if a vector field $V : \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz continuous, then the ODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = V(X_t), \quad X_0 = x \in \mathbb{R}^d \tag{1}$$

determines a unique flow of homeomorphisms on $\mathbb{R}^d$, such that the Lebesgue measure $\mathcal{L}_d$ is quasi-invariant, and the Radon-Nikodym density

$$\rho_t(x) := \frac{\mathrm{d}(\mathcal{L}_d \circ X_t^{-1})}{\mathrm{d}\mathcal{L}_d}(x) = \exp\left(-\int_0^t \mathrm{div}(V)\big[X_s(X_t^{-1}(x))\big]\,\mathrm{d}s \right).$$

Di Perna–Lions theory

In applications (e.g. in kinetic theory and fluid mechanics), the vector fields V often have only Sobolev or even BV regularity.

Many people have tried to generalize the Cauchy–Lipschitz theory to the cases where V is not regular. A breakthrough was made by Di Perna and Lions (1989).

- Di Perna–Lions (Invent. Math., 1989): If

$$V \in W^{1,1}_{loc}, \ \frac{|V|}{1+|x|} \in L^1 \cap L^\infty \text{ and } \mathrm{div}(V) \in L^\infty, \qquad (2)$$

  then ODE (1) generates a unique flow of measurable maps $X_t : \mathbb{R}^d \to \mathbb{R}^d$, such that the Lebesgue measure λ is quasi-invariant under the action of X_t.

Their approach is indirect, and it is in fact an extension of the classical characteristics method.

Remarks on the DiPerna–Lions flow X_t

Remark 1

- *Note that the solution X_t is not required to be well defined for each starting point $x \in \mathbb{R}^d$. The map $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is not continuous, but only measurable.*
- *The quasi-invariance of X_t implies that it does not map a set $E \subset \mathbb{R}^d$ of positive Lebesgue measure to a negligible set (otherwise the composition $V(X_t)$ makes no sense).*

Weak differentiability of DiPerna–Lions flow X_t: (i)

But X_t is weakly differentiable in a certain sense.

- Le Bris–Lions (2004) proved the differentiability in measure of X_t. More precisely, they considered

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X_t(x) = V(X_t(x)), & X_0(x) = x; \\ \frac{\mathrm{d}}{\mathrm{d}t} Y_t(x,y) = [\nabla V(X_t(x))] Y_t(x,y), & Y_0(x,y) = y \end{cases}$$

and for $\varepsilon > 0$,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X_t(x) = V(X_t(x)), & X_0(x) = x; \\ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{X_t(x+\varepsilon y) - X_t(x)}{\varepsilon} \right] = \frac{V(X_t(x+\varepsilon y)) - V(X_t(x))}{\varepsilon}, & \frac{X_0(x+\varepsilon y) - X_0(x)}{\varepsilon} = y. \end{cases}$$

Then under (2), they proved that as $\varepsilon \downarrow 0$,

$$\frac{X_t(x+\varepsilon y) - X_t(x)}{\varepsilon} \to Y_t(x,y) \quad \text{locally in measure in } \mathbb{R}^d \times \mathbb{R}^d.$$

Weak differentiability of DiPerna–Lions flow X_t: (ii)

- When $V \in W^{1,1}_{loc}$ and $\nabla V \in (L^1 \log L^1)_{loc}$, then Ambrosio et al. (2005) proved that $X_t : \mathbb{R}^n \to \mathbb{R}^n$ is approximately differentiable.
- Crippa–De Lellis (2008) proved the same result by estimating the following quantity

$$A(R,\lambda) := \int_{B_R(0)\cap G_\lambda} \left[\sup_{0\le t\le T} \sup_{0<r<2R} \fint_{B_r(x)\cap G_\lambda} \log\left(\frac{|X_t(x)-X_t(y)|}{r} + 1 \right) \mathrm{d}y \right] \mathrm{d}x \tag{3}$$

in terms of R and $\int_{B_{3\lambda}(0)} |\nabla V| \log |\nabla V| \, \mathrm{d}x$, where

- $B_r(x) = \{y : |x-y| \le r\}$;
- $G_\lambda = \{x \in \mathbb{R}^n : \sup_{t\in[0,T]} |X_t(x)| \le \lambda\}$ is the level set.

Generalized stochastic flow of Itô's SDE

Let $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}$ be measurable and B_t is an m-dimensional standard B.M. defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + V(X_t)\,\mathrm{d}t, \quad X_0 = x. \tag{4}$$

Fix a (finite) reference measure μ on $\mathbb{R}^d$.
A measurable map $X : \Omega \times \mathbb{R}^d \to C([0,T], \mathbb{R}^d)$ is called the generalized stochastic flow associated to (4) if

(i) $\forall\, t \in [0,T]$ and a.e. $x \in \mathbb{R}^d$, $X_t(\cdot, x) : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_t := \sigma(B_s : s \le t)$-measurable;

(ii) $\forall\, t \in [0,T]$, $\exists\, \rho_t \in L^1(\mathbb{P} \times \mu)$ s.t. $(X_t(\omega, \cdot))_{\#}\mu = \rho_t(\omega, \cdot)\mu$;

(iii) μ-a.e. $x \in \mathbb{R}^d$, the following equality holds $\mathbb{P}$-a.s.:

$$X_t(\omega, x) = x + \int_0^t \sigma(X_s(\omega, x))\,\mathrm{d}B_s + \int_0^t V(X_s(\omega, x))\,\mathrm{d}s, \ \forall\, t \in [0,T];$$

(iv) flow property: $X_{t+s}(\omega, x) = X_t(\theta_s\omega, X_s(\omega, x))$.

Known results on SDE (4)

DiPerna–Lions's original approach does not work for SDE, since the related stochastic transport equation is always degenerate.

X. Zhang (BSM, 2010) first applied Crippa–De Lellis's direct method to study SDE with Sobolev coefficients.

- X. Zhang (BSM, 2010): $\frac{|V|}{1+|x|}, \mathrm{div}(V) \in L^\infty$ and one of the conditions below holds:
    - $V \in BV_{loc}, \sigma$ is constant (basically reduced to ODE);
    - $|\nabla V| \in (L^1 \log L^1)_{loc}$ and $|\nabla \sigma|, \big(\sup_{|z|\le 1} |\sigma(\cdot - z)|\big)|\nabla \mathrm{div}(\sigma)| \in L^\infty$.

  Then SDE (4) generates a stochastic flow X_t of measurable maps which leaves the Lebesgue measure $\mathcal{L}_d$ absolutely continuous.

Known results on SDE (4): cont.

- Fang-Luo-Thalmaier (JFA, 2010): Take standard Gaussian γ_d as reference measure. Assume
  - $\sigma \in \cap_{q>1}\mathbb{D}_1^q(\gamma_d)$ and $V \in \mathbb{D}_1^{q_0}(\gamma_d)$ for some $q_0 > 1$;
  - σ, V have linear growth;
  - $\exists\, \lambda_0 > 0$ such that $\int_{\mathbb{R}^d} \exp\left[\lambda_0\big(|\mathrm{div}_{\gamma_d}(V)| + |\nabla\sigma|^2 + |\mathrm{div}_{\gamma_d}(\sigma)|^2\big)\right] \mathrm{d}\gamma_d < +\infty$.

  Then SDE (4) generates a unique generalized stochastic flow X_t and the Radon-Nikodym density $\rho_t \in L^1 \log L^1(\mathbb{P} \times \gamma_d)$.
- X. Zhang (to appear): Cauchy measure $\mathrm{d}\mu = (1+|x|^2)^{-\alpha}\,\mathrm{d}x$ for some $\alpha > d/2$. Assume
  - $\sigma \in W^{1,2q}_{loc}, V \in W^{1,q}_{loc}$ for some $q > 1$;
  - σ, V have linear growth;
  - $\forall\, p \geq 1$, it holds $\int_{\mathbb{R}^d} \exp\left[p\big([\mathrm{div}(V)]^- + |\nabla\sigma|^2\big)\right] \mathrm{d}\mu < +\infty$.

  Then similar results hold.

Known results on SDE (4): cont.

- Luo (recent work): Fix $q > 1$ and let $\mathrm{d}\mu = (1+|x|^2)^{-\alpha}\,\mathrm{d}x$ for some $\alpha > q + d/2$. Assume
  - $\sigma \in W^{1,2q}_{loc}$, $V \in W^{1,q}_{loc}$;
  - $\exists\, p_0 > 0$ such that $\int_{\mathbb{R}^d} \exp\Big\{p_0\Big([\mathrm{div}(V)]^- + \frac{|V|}{1+|x|} + \big(\frac{|\sigma|}{1+|x|}\big)^2 + |\nabla\sigma|^2\Big)\Big\}\,\mathrm{d}\mu < +\infty.$

  Then there is a unique generalized stochastic flow X_t associated to SDE (4) and the Radon-Nikodym density $\rho_t \in L^1 \log L^1(\mathbb{P} \times \mu)$.
  The coefficients σ and V do not necessarily have linear growth and they may be locally unbounded.

Therefore the existence and uniqueness of generalized stochastic flows have been well established.

It remains to study their regularity.

Purpose of this talk

Our purpose is to show the weak differentiability of the stochastic flow of maps $X_t : \mathbb{R}^d \to \mathbb{R}^d$.

This time, however, I am unable to follow Crippa–De Lellis's method to show the approximate differentiability of X_t.

The ideal result would be: for $\mathbb{P}$-a.s. $\omega \in \Omega$, $\forall\, t \geq 0$, $X_t(\omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is approximately differentiable.

It seems that a stochastic analogue of (3) does not imply the above property.

Therefore, we turn to Le Bris–Lions's approach by considering

$$\begin{cases} \mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + V(X_t(x))\,\mathrm{d}t,\ X_0(x) = x, \\ \mathrm{d}Y_t(x,y) = \big[\nabla\sigma(X_t(x))\big]\,Y_t(x,y)\,\mathrm{d}B_t \\ \qquad\qquad + \big[\nabla V(X_t(x))\big]\,Y_t(x,y)\,\mathrm{d}t,\ Y_0(x,y) = y. \end{cases} \tag{5}$$

and for $\varepsilon > 0$,

$$\begin{cases} \mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + V(X_t(x))\,\mathrm{d}t,\ X_0(x) = x, \\ \mathrm{d}\Big[\frac{X_t(x+\varepsilon y)-X_t(x)}{\varepsilon}\Big] = \frac{\sigma(X_t(x+\varepsilon y))-\sigma(X_t(x))}{\varepsilon}\,\mathrm{d}B_t \\ \qquad + \frac{b(X_t(x+\varepsilon y))-b(X_t(x))}{\varepsilon}\,\mathrm{d}t,\ \ \frac{X_0(x+\varepsilon y)-X_0(x)}{\varepsilon} = y. \end{cases} \tag{6}$$

We want to show that both systems (5) and (6) generate flows $Z_t(x,y) = (X_t(x), Y_t(x,y))$, $\quad Z_t^\varepsilon(x,y) = \Big(X_t(x), \frac{X_t(x+\varepsilon y)-X_t(x)}{\varepsilon}\Big)$; moreover, as $\varepsilon \downarrow 0$,

$$\frac{X_t(x+\varepsilon y) - X_t(x)}{\varepsilon} \to Y_t(x,y) \quad \text{in some sense.}$$

Both systems (5) and (6) can be interpreted as special cases of the following SDE with partially Sobolev coefficients:

$$\begin{cases} \mathrm{d}X_{1,t} = \sigma_1(X_{1,t})\,\mathrm{d}B_t + V_1(X_{1,t})\,\mathrm{d}t, \ X_{1,0} = x_1, \\ \mathrm{d}X_{2,t} = \sigma_2(X_{1,t}, X_{2,t})\,\mathrm{d}B_t + V_2(X_{1,t}, X_{2,t})\,\mathrm{d}t, \ X_{2,0} = x_2, \end{cases} \tag{7}$$

where $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$ and

- $\sigma_1 : \mathbb{R}^{n_1} \to \mathcal{M}_{n_1,m}, \ V_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1};$
- $\sigma_2 : \mathbb{R}^{n} \to \mathcal{M}_{n_2,m}, \ V_2 : \mathbb{R}^{n} \to \mathbb{R}^{n_2}.$

Take the system (6) for example: set

- $n_1 = n_2 = d, \ x_1 = x, \ x_2 = y,$
- $X_{1,t} = X_t, \ X^{\varepsilon}_{2,t} = \frac{X_t(x+\varepsilon y) - X_t(x)}{\varepsilon},$
- $\sigma_1 = \sigma, \ V_1 = V,$
- $\sigma^{\varepsilon}_2 = \frac{\sigma(x+\varepsilon y) - \sigma(x)}{\varepsilon}, V^{\varepsilon}_2 = \frac{V(x+\varepsilon y) - V(x)}{\varepsilon}.$

Outline

Assumptions

In this section we consider the systems of SDEs (7).

Fix some $q > 1$ and choose $\alpha_1 > q + n_1/2$, $\alpha > \alpha_1 + n_2/2$. Let

$$\mathrm{d}\mu_1(x_1) = (1 + |x_1|^2)^{-\alpha_1}\,\mathrm{d}x_1 \quad \text{and} \quad \mathrm{d}\mu(x) = (1 + |x|^2)^{-\alpha}\,\mathrm{d}x.$$

Then $\mu_1(\mathbb{R}^{n_1}) < \infty$ and $\mu(\mathbb{R}^n) < \infty$.

Our assumptions are:

- Conditions on σ_1 and V_1:
  - (H1) $\sigma_1 \in W^{1,2q}_{x_1,loc}$, $V_1 \in W^{1,q}_{x_1,loc}$;
  - (H2) $\int_{\mathbb{R}^{n_1}} \exp\Big[p_0\Big([\mathrm{div}_{x_1}(V_1)]^- + \frac{|V_1|}{1+|x_1|} + \Big(\frac{|\sigma_1|}{1+|x_1|}\Big)^2 + |\nabla_{x_1}\sigma_1|^2\Big)\Big]\mathrm{d}\mu_1 < +\infty$ for some $p_0 > 0$;
- Conditions on σ_2 and V_2:
  - (H3) $\sigma_2 \in L^{2q}_{x_1,loc}(W^{1,2q}_{x_2,loc})$, $V_2 \in L^{q}_{x_1,loc}(W^{1,q}_{x_2,loc})$;
  - (H4) $\int_{\mathbb{R}^{n}} \exp\Big[p_0\Big([\mathrm{div}_{x_2}(V_2)]^- + \frac{|V_2|}{1+|x|} + \Big(\frac{|\sigma_2|}{1+|x|}\Big)^2 + |\nabla_{x_2}\sigma_2|^2\Big)\Big]\mathrm{d}\mu < +\infty$ for some $p_0 > 0$.

Main result

Theorem 2 (Existence and Uniqueness)

Under the assumptions (H1)–(H4), the Itô SDE (7) *generates a unique stochastic flow* $X_t = (X_{1,t}, X_{2,t})$ *such that the reference measure* μ *is absolutely continuous.*

Moreover, the Radon–Nikodym density $\rho_t := \frac{\mathrm{d}[(X_t)_{\#}\mu]}{\mathrm{d}\mu}$ *satisfies* $\rho_t \in L^1 \log L^1(\mathbb{P} \times \mu)$.

Ideas of the proof

Suppose we have a sequence of "smooth" flows X_t^k with $(X_t^k)_\# \mu = \rho_t^k \mu$, $k \geq 1$. For $f \in C_c(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f(X_t^k(x))\, \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} f(y)\, \rho_t^k(y)\, \mathrm{d}\mu(y).$$

Multiply by a random variable $\xi \in L^\infty(\Omega)$ and take expectation,

$$\begin{array}{ccc}
\mathbb{E} \displaystyle\int_{\mathbb{R}^n} \xi(\cdot)\, f(X_t^k(x))\, \mathrm{d}\mu(x) & = & \mathbb{E} \displaystyle\int_{\mathbb{R}^n} \xi(\cdot)\ f(y)\, \rho_t^k(y)\, \mathrm{d}\mu(y) \\
(2) \Big\downarrow & & \Big\downarrow (1) \\
\mathbb{E} \displaystyle\int_{\mathbb{R}^n} \xi(\cdot)\, f(X_t(x))\, \mathrm{d}\mu(x) & = & \mathbb{E} \displaystyle\int_{\mathbb{R}^n} \xi(\cdot)\ f(y)\, \rho_t(y)\, \mathrm{d}\mu(y)
\end{array}$$

Then by the arbitrariness of ξ and f, we have $(X_t)_\# \mu = \rho_t \mu$.

First ingredient: a-priori estimate of R–N density

Suppose $\sigma \in C_b^\infty(\mathbb{R}^n, \mathcal{M}_{n,m})$ and $V \in C_b^\infty(\mathbb{R}^n, \mathbb{R}^n)$; then SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + V(X_t)\,\mathrm{d}t, \quad X_0 = x$$

generates a unique stochastic flow of diffeomorphisms on $\mathbb{R}^n$.

Take a finite reference measure $\mathrm{d}\mu = e^{\lambda(x)}\,\mathrm{d}x$ where $\lambda \in C^2(\mathbb{R}^n)$. Then a.s., $\forall\, t > 0$, the push-forward $(X_t)_\#\mu \ll \mu$. Define

$$\rho_t(x) = \frac{\mathrm{d}[(X_t)_\#\mu]}{\mathrm{d}\mu}(x).$$

X. Zhang (2012, to appear) proved the following estimate for ρ_t: $\forall\, t \in [0, T]$, $p > 1$,

$$\|\rho_t\|_{L^p(\mathbb{P}\times\mu)} \le \mu(\mathbb{R}^n)^{\frac{1}{p+1}} \left[\sup_{t\in[0,T]} \int_{\mathbb{R}^n} e^{tp^2(p|\Lambda_1^\sigma|^2 - \Lambda_2^{\sigma,V})}\,\mathrm{d}\mu \right]^{\frac{1}{p(p+1)}}. \quad (8)$$

First ingredient: a-priori estimate of R–N density (cont.)

In the above formula, $\Lambda_1^\sigma = \mathrm{div}(\sigma) + \sigma^* \nabla\lambda$ and

$$\Lambda_2^{\sigma,V} = \mathrm{div}(V) + \frac{1}{2}\langle \sigma\sigma^*, \mathrm{Hess}(\lambda)\rangle + \langle V, \nabla\lambda\rangle - \frac{1}{2}\langle \nabla\sigma, (\nabla\sigma)^*\rangle.$$

Rewriting the last term in component form,

$$\langle \nabla\sigma, (\nabla\sigma)^*\rangle = \sum_{k=1}^m \langle \nabla\sigma^{\cdot k}, (\nabla\sigma^{\cdot k})^*\rangle = \sum_{k=1}^m \sum_{i,j=1}^n (\partial_i \sigma^{jk})(\partial_j \sigma^{ik}).$$

Key observation: if the first n_1-rows $\sigma_1 := (\sigma^{ij})_{1\le i\le n_1, 1\le j\le m}$ only depend on the variables $x_1 = (x^1, \ldots, x^{n_1})$, then

$$\begin{aligned}\langle \nabla\sigma, (\nabla\sigma)^*\rangle &= \sum_{k=1}^m \Bigg[\sum_{i,j=1}^{n_1} (\partial_i \sigma^{jk})(\partial_j \sigma^{ik}) + \sum_{i,j=n_1+1}^{n} (\partial_i \sigma^{jk})(\partial_j \sigma^{ik})\Bigg] \\ &= \langle \nabla_{x_1}\sigma_1, (\nabla_{x_1}\sigma_1)^*\rangle + \langle \nabla_{x_2}\sigma_2, (\nabla_{x_2}\sigma_2)^*\rangle.\end{aligned}$$

First ingredient: a-priori estimate of R–N density (cont.)

Regularize the coefficients σ_1, V_1 (resp. σ_2, V_2) to obtain $\sigma_{1,k}$, $V_{1,k}$ (resp. $\sigma_{2,k}$, $V_{2,k}$).

Denote the corresponding smooth solution flows by $X_t^k = (X_{1,t}^k, X_{2,t}^k)$ and $\rho_t^k = \frac{\mathrm{d}[(X_t^k)_{\#}\mu]}{\mathrm{d}\mu}$.

Proposition 3 (Uniform density estimate)

For fixed $p > 1$, there are two positive constants $C_{1,p}, C_{2,p} > 0$ and $T_0 > 0$ small enough such that $\forall\, 0 \le t \le T_0$,

$$\|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} \le$$

$$C_{1,p}\left(\int_{\mathbb{R}^{n_1}} e^{C_{2,p}T_0\left([\mathrm{div}_{x_1}(V_1)]^- + \frac{|V_1|}{1+|x_1|} + \left(\frac{|\sigma_1|}{1+|x_1|}\right)^2 + |\nabla_{x_1}\sigma_1|^2\right)}\mathrm{d}\mu_1 \right.$$

$$\left. \times \int_{\mathbb{R}^n} e^{C_{2,p}T_0\left([\mathrm{div}_{x_2}(V_2)]^- + \frac{|V_2|}{1+|x|} + \left(\frac{|\sigma_2|}{1+|x|}\right)^2 + |\nabla_{x_2}\sigma_2|^2\right)}\mathrm{d}\mu\right)^{\frac{1}{p(p+1)}}.$$

Second ingredient: stability

Proposition 4 (Stability)

Consider

$$\begin{cases} \mathrm{d}X_{1,t} = \sigma_1(X_{1,t})\,\mathrm{d}B_t + V_1(X_{1,t})\,\mathrm{d}t, & X_{1,0} = x_1, \\ \mathrm{d}X_{2,t}^k = \sigma_{2,k}(X_{1,t}, X_{2,t}^k)\,\mathrm{d}B_t + V_{2,k}(X_{1,t}, X_{2,t}^k)\,\mathrm{d}t, & X_{2,0}^k = x_2. \end{cases}$$

where

- σ_1, V_1 *verify (H1), (H2),* $\sigma_{2,k}, V_{2,k}$ *verify (H3), (H4),*
- $\sigma_{2,k} \xrightarrow{L^{2q}_{loc}(\mathbb{R}^n)} \sigma_2,\ V_{2,k} \xrightarrow{L^{q}_{loc}(\mathbb{R}^n)} V_2$ *as* $k \to \infty$.

We also assume that

$$C_1 := \sup_{k\geq 1} \left[\|\sigma_{2,k}\|_{L^{2q}(\mu)} + \|b_{2,k}\|_{L^q(\mu)}\right] < +\infty,$$

Second ingredient: stability (cont.)

Proposition 4 (Stability (cont.))

and for any $R > 0$,

$$C_{2,R} := \sup_{k\geq 1} \left[\|\nabla_{x_2} b_{2,k}\|_{L^q(B(R))} + \|\nabla_{x_2} \sigma_{2,k}\|_{L^{2q}(B(R))} \right] < +\infty.$$

Suppose that for all $k \geq 1$*, the density function* $\rho_t^k := \frac{\mathrm{d}(X_t^k)_{\#}\mu}{\mathrm{d}\mu}$ *exists and*

$$\Lambda_{p,T} := \sup_{k\geq 1} \sup_{0\leq t\leq T} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} < +\infty.$$

Then there exists a random field $X_2 : \Omega \times \mathbb{R}^n \to C([0,T], \mathbb{R}^{n_2})$ *such that*

$$\lim_{k\to\infty} \mathbb{E} \int_{\mathbb{R}^n} 1 \wedge \|X_2^k - X_2\|_{\infty,T} \, \mathrm{d}\mu = 0.$$

Now Theorem 2 follows from Propositions 3 and 4.

Outline

Now we come back to the Itô SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + V(X_t)\,\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d. \tag{4}$$

We want to prove the differentiability of $X_t : \mathbb{R}^d \to \mathbb{R}^d$ under suitable conditions. Hence we return to the two systems (5), (6):

$$\begin{cases} \mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + V(X_t(x))\,\mathrm{d}t, \ X_0(x) = x, \\ \mathrm{d}Y_t(x,y) = \big[\nabla\sigma(X_t(x))\big]\,Y_t(x,y)\,\mathrm{d}B_t \\ \qquad\qquad + \big[\nabla V(X_t(x))\big]\,Y_t(x,y)\,\mathrm{d}t, \ Y_0(x,y) = y \end{cases} \tag{5}$$

and for $\varepsilon > 0$,

$$\begin{cases} \mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + V(X_t(x))\,\mathrm{d}t, \ X_0(x) = x, \\ \mathrm{d}\Big[\frac{X_t(x+\varepsilon y) - X_t(x)}{\varepsilon}\Big] = \frac{\sigma(X_t(x+\varepsilon y)) - \sigma(X_t(x))}{\varepsilon}\,\mathrm{d}B_t \\ \qquad + \frac{b(X_t(x+\varepsilon y)) - b(X_t(x))}{\varepsilon}\,\mathrm{d}t, \ \frac{X_0(x+\varepsilon y) - X_0(x)}{\varepsilon} = y. \end{cases} \tag{6}$$

We fix $q > 1$, $\alpha_1 > q + d/2$ and $\alpha > 2\alpha_1 + q + d/2$. Let

$$\mathrm{d}\mu_1 = (1 + |x|^2)^{-\alpha_1}\,\mathrm{d}x, \quad \mathrm{d}\mu = (1 + |x|^2 + |y|^2)^{-\alpha}\,\mathrm{d}x\mathrm{d}y.$$

Lemma 5

Assume

(A1) $\sigma \in W^{1,2q}_{loc}$ *and* $b \in W^{1,q}_{loc}$;

(A2) $\int_{\mathbb{R}^d} \exp\left[p_0\left(|\nabla V| + \frac{|V|}{1+|x|} + \left(\frac{|\sigma|}{1+|x|}\right)^2 + |\nabla\sigma|^2\right)\right]\mathrm{d}\mu < +\infty$ *for some* $p_0 > 0$.

Then both systems (5) *and* (6) *verify the conditions (H1)–(H4).*

Therefore we can apply the stability result (Proposition 4) to show

Theorem 6

Under (A1) and (A2), we have for all $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\int_{\mathbb{R}^{2d}} 1 \wedge \left(\sup_{0\le t\le T} \|Y^\varepsilon_t - Y_t\|\right)\mathrm{d}\mu = 0.$$

Thank you for your attention!