On time regularity of generalized Ornstein-Uhlenbeck processes with cylindrical stable noise

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Outline

- Problems
- Main Results
- Proofs
- Some Discussions

1 Problems

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0. \tag{1}$$

H, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$.

A, generator of a C_0-semigroup on H, A^* the adjoint operator of A.

L, Lévy process, $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$,

L^n, i.i.d., càdlàg real-valued Lévy processes.

$\{e_n\}_{n \in \mathbb{N}}$, fixed reference orthonormal basis in H.

β_n, a sequence of positive numbers.

Problem :

If the solution of Eq. (1) $(X(t))_{t\geq 0}$ takes value in H for any t, is there a H-valued càdlàg modification of X? i.e. $\exists$? a H-valued càdlàg $(\tilde{X}_t)_{t\geq 0}$ such that,

$$\mathbb{P}(X_t = \tilde{X}_t) = 1, \quad \text{for any } \ t. \tag{2}$$

Assume that $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A^*)$, the weak solution of Eq. (1),

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0.$$

can be represented by for any $n \in \mathbb{N}$,

$$d\langle X(t), e_n\rangle_H = \langle X(t), A^* e_n\rangle_H dt + \beta_n dL^n(t). \tag{3}$$

$\langle X(t), e_n\rangle_H \equiv X^n(t)$.

L^n, $\alpha-$stable processes, $\alpha \in (0, 2)$.

1.1. Property of Sample Paths

Kolmogorov's Extension Theorem:

$\mathcal{S}$: State space.

construct distribution on $\mathcal{S}^{[0,\infty)}$.

However, this theorem does not describe the properties of sample paths.

Continuous or càdlàg modification of sample path is a fundamental property in Theory of Stochastic Processes, such as Martingale Theory, Markov Processes and Probabilistic Potential Theory and SDE.

[1] Doob, J.L. *Stochastic Processes.* John Wiley & Sons Inc., New York 1953

1.2. Generalized Ornstein-Uhlenbeck Processes

$$dX(t) = AX(t)dt + dL(t).$$

$L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n i.i.d., càdlàg α-stable processes.

Modeling some heavy tail phenomenon.

The time regularity of the process X is of prime interest in the study of non-linear Stochastic PDEs.

And these studies of generalized O-U processes is a beginning point.

1.3. l^2-valued O-U processes driven by Brownian motion

- l^2-valued O-U processes driven by Brownian motion

  [2] Iscoe, Marcus, McDonald, Talagrand, Zinn, (1990) *Ann. Proba.*

$$dx_k(t) = -\lambda_k x_k(t)dt + \sqrt{2a_k}dB_k, \quad k = 1, 2, \cdots .$$

They gave a simple but quite sharp criterion for continuity of X_t in l^2.

Theorem 1 in [2] *$f(x)$ positive function on $[0,\infty)$ such that $\frac{f(x)}{x}$ non-decreasing for $x \geq x_1 > 0$ and*

$$\int_{x_1}^{\infty} \frac{dx}{f(x)} < \infty, \quad \sum_k \frac{a_k}{\lambda_k} < \infty, \quad \sup_k \frac{f(a_k) \vee x_1}{\lambda_k \vee 1} < \infty. \tag{4}$$

Then, x_t is continuous in l^2 a.s. Moreover, this result is best possible in the sense that it is false for any function $f(x)$, which satisfies all the above hypotheses with the exception that $\int_{x_1}^{\infty} \frac{dx}{f(x)} = \infty$.

- H or B-valued O-U processes

  [3] Millet, Smolenski (1992) *Prob. Theory Related Fields.*

1.3.1. O-U Eq. with Lévy noise

[4] Fuhrman, Röckner (2000) *Generalized Mahler semigroups: the non Gaussian case,* Potential Anal., 12(2000), 1-47.

- *There is an enlarged space E, $H \subset_{\mathrm{HS}} E$, such that $(X(t))_{t\geq 0}$ has a càdlàg path in E.*

[5] Priola, E., Zabczyk, J. *On linear evolution with cylindrical Lévy noise,*in: SPDE and Applications VIII, Proceedings of the Levico 2008 Conference.

- *$L(t)$ symmetric, and $L(t) \in U \supset H$, they give a necessary and sufficient condition of $X_t \in H$, for any $t > 0$.*

[6] Brzeźniak, Z., Zabczyk, J. *Regularity of Ornstein-Uhlenbeck processes driven by Lévy white noise,* Potential Anal. 32(2010)153-188.

- *$L(t)$, Lévy white noise obtained by subordination of a Gaussian white noise. $L_t = W(Z(t))$, Spatial continuity, Time irregularity.*

[7] Priola, E., Zabczyk, J. *Structural properties of semilinear SPDEs driven by cylindrical stable process,* Probab. Theory Related Fields, 149(2011), 97-137 [PZ11]

- *They conjectured in Section 4 in [7], If L^n are symmetric α-stable processes, $\alpha \in (0,2)$, the H-càdlàg property of Eq. (1) holds under much weaker conditions than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty$.*

Remark 1. $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Leftrightarrow L(t) = \sum_{n=0}^{\infty} \beta_n L^n(t) e_n$ has H-càdlàg property.

Remark 2. In general, $L \in H \Rightarrow X$ has H-càdlàg path.

[8] Brzeźniak, Z., Goldys, B., Imkeller, P., Peszat, S., Priola, E., Zabczyk, J. *Time irregularity of generalized Ornstein-Uhlenbeck processes*,C. R. Acad. Sci. Paris, Ser. I 348(2010), 273-276. [BGIPPZ10]

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0.$$

$$d\langle X(t), e_n\rangle_H = \langle X(t), A^* e_n\rangle_H dt + \beta_n dL^n(t),\ n \in \mathbb{N}. \tag{5}$$

$\langle X(t), e_n\rangle_H \equiv X^n(t)$.

- **Theorem 2.1 [8]** X, H-valued process, $(e_n) \in \mathscr{D}(A^*)$, $\beta_n \not\to 0$, then X has no H-càdlàg modification with probability 1.

- Question 1,2,3,4 ... ...

[9] Brzeźniak, Z., Otobe, Y. and Xie B. *Regularity for SPDE driven by α-stable cylindrical noise. 2011, preprint*

- They obtained detailed results of spatial regularity and temporal integrability.

2 Main Results

$L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, L^n i.i.d. real-valued Lévy processes, Lévy characteristic measure ν. $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A^*)$,

$$d\langle X(t), e_n\rangle_H = \langle X(t), A^* e_n\rangle_H dt + \beta_n dL^n(t).$$

Theorem 1 *Assume that the process X in Eq. (1) has H-càdlàg modification, then for any $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} \nu(|y| \geq \epsilon/\beta_n) < \infty.$$

Remark 3. This theorem implies Theorem 2.1 in [BGIPPZ10]

$\beta_n \nrightarrow 0 \Rightarrow$ no H-càdlàg modification

L^n, i.i.d. α-stable process. $\nu(dy) = \begin{cases} c_1 y^{-1-\alpha} dy, & y > 0, \\ c_2 |y|^{-1-\alpha} dy, & y < 0. \end{cases}$

Theorem 2 *Assume* $(L^n, n = 1, 2, \cdots)$ *are i.i.d., non-trivial* α*-stable processes,* $\alpha \in (0, 2)$*, and* $S(t) = e^{At}$ *satisfying* $\|S(t)\|_{L(H,H)} \leq e^{\beta t}$, $\beta \geq 0$, *(generalized contraction principle), the following three assertions are equivalent:*

(1) the process $(X(t), t \geq 0)$ *in Eq. (1) has* H*-càdlàg modification;*

(2) $\sum_{n=1}^{\infty} |\beta_n|^\alpha < \infty$*;*

(3) the process L *is a Lévy process on* H*.*

Remark 4. This result denies the conjecture in [PZ11]. And more, Theorem 2 does not need the assumption of symmetry of L_n.

much weaker than $\sum_{n=1}^{\infty} |\beta_n|^\alpha < \infty$.

Remark 5. In [BGIPPZ10],

Question 3: *Is the requirement of the process L evolves in H also necessary for the existence of H-càdlàg modification of X?*

Theorem 2 partly answers Question 3, *i.e.* at least if L^n, i.i.d. α-stable processes, L evolving in H is a necessary condition of X having H-càdlàg modification.

Moreover, if A is self-adjoint, eigenvectors e_n, eigenvalues $-\lambda_n < 0$, $n \in \mathbb{N}$,

$$dX^n(t) = -\lambda_n X^n(t)dt + \beta_n dL^n(t), \ \ t \geq 0, \ \ n \in \mathbb{N}. \tag{6}$$

For $\delta \in \mathbb{R}$,

$$H_\delta \equiv \mathscr{D}(A^{\delta/2}) = \left\{x = \sum_{n=1}^{\infty} x_n e_n : \ \sum_{n=1}^{\infty} \lambda_n^\delta |x_n|^2 < \infty, \ x_n \in \mathbb{R}\right\}.$$

Proposition 3 *Assume L^n are i.i.d., non-trivial α-stable processes, $\alpha \in (0, 2)$ and X^n is the solution of Eq. (6). Then the following assertions are equivalent:*

(1) the process $(X(t), t \geq 0)$ in Eq. (1) has H_δ-$c\grave{a}dl\grave{a}g$ modification;

(2) $\sum_{n=1}^{\infty} |\beta_n \lambda_n^{\delta/2}|^\alpha < \infty$;

(3) the process L is a Lévy process on H_δ.

Furthermore, we apply Proposition 3 to Stochastic Heat Equation (S.H.E.) on $\mathcal{O} = (0, \pi)$ with α-stable noise

$$dX(t) = \Delta X(t)dt + dL(t), \tag{7}$$

Proposition 4 *If $\beta_n = 1$ for any $n \in \mathbb{N}$, Eq. (7) has H_δ-càdlàg modification if and only if $\delta < -1/\alpha$.*

Remark 6. in [BGIPPZ10]

Question 4: *Is the process X in S.H.E. H_δ-càdlàg for $\delta \in [-\frac{1}{\alpha}, 0)$?*

Proposition 4 answers Question 4.

Proposition 5 *Assume L^n are i.i.d., non-trivial symmetric α-stable processes. If $(\beta_n, n \geq 1)$ satisfies $\sum_{n=1}^{\infty} \beta_n^{\alpha}/n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n^{\alpha} = \infty$, then there is no H-càdlàg modification of $(X(t), t \geq 0)$ in Eq. (7), even if for any $t > 0$, $X(t) \in H$.*

Remark 7. In [BGIPPZ10],

Question 1: *Does $\beta_n \to 0$ imply existence of a càdlàg modification of X?*

If we set $\beta_n = n^{-\frac{1}{\alpha}}$, then $\sum_{n=1}^{\infty} \beta_n^{\alpha}/n^2 < \infty$, $\sum_{n=1}^{\infty} \beta_n^{\alpha} = \infty$ and $\beta_n \to 0$ in Eq. (7) (S.H.E.). By Proposition 5, we give an example showing that $\beta_n \to 0$ does not imply the existence of H-càdlàg modification of X, even if for any $t > 0$, $X(t) \in H$ and the Lévy characteristic measure of L supports on H. This is a negative answer to Question 1 .

Remark 8. Question 2 in [BGIPPZ10]: Is $e_n \in \mathscr{D}(A^*)$ essential for the validity of Theorem 2.1 .

We have no idea to this question.

3 Proofs

$$X(t) = \sum_{n=1}^{\infty} X^n(t) e_n, \quad X^n(t) = \langle X(t), e_n \rangle_H$$

Lemma 1 *The process* $(X(t), t \geq 0)$ *is a* H*-càdlàg (resp. continuous) process with probability 1, if and only if for any* $n \in \mathbb{N}$*, the process* $(X^n(t), t \geq 0)$ *is càdlàg (resp. continuous) process with probability 1 and for any* $T > 0$*,*

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \sum_{i=N}^{\infty} |X^i(t)|^2 = 0, \quad \textit{with probability 1}. \tag{8}$$

Set $\triangle f(t) = f(t) - f(t-)$. Noting that if $(X(t), t \geq 0)$ is a H-càdlàg process, then

$$\sup_{n \geq N} \sup_{t \in [0,T]} |\triangle X^n(t)| \leq 2\Big(\sup_{t \in [0,T]} \sum_{n=N}^{\infty} |X^n(t)|^2 \Big)^{1/2}$$

Lemma 2 *Assume the process $(X(t), t \geq 0)$ is a H-càdlàg process with probability 1, then for any $T > 0$,*

$$\lim_{N \to \infty} \sup_{n \geq N} \sup_{t \in [0,T]} |\triangle X^n(t)| = 0, \text{ with probability 1.}$$

Proof of Theorem 1 X, H-càdlàg property.

$$\tau_n = \inf\{t > 0 : |\beta_n \triangle L^n(t)| \geq \epsilon\}$$

τ_n independent exponential distributions with parameter $\psi_n = \nu(|y| \geq \epsilon/\beta_n)$.

Lemma 2 implies

$$\lim_{N\to\infty} \mathbb{P}\big(\tau_n \leq T, \text{ for some } n \geq N\big) = 0.$$

$$\mathbb{P}\big(\tau_n \leq T, \text{ for some } n \geq N\big) = 1 - \prod_{n\geq N} \mathbb{P}\big(\tau_n \leq T\big) = 1 - \exp\Big(-\sum_{n=N}^{\infty} \psi_n T\Big)$$

$$\sum_{n=1}^{\infty} \nu(|y| \geq \epsilon/\beta_n) = \sum_{n=1}^{\infty} \psi_n < \infty.$$

Applying Theorem 1 to α-stable processes,

$$\nu(dy) = \begin{cases} c_1 y^{-1-\alpha} dy, & y > 0, \\ c_2 |y|^{-1-\alpha} dy, & y < 0. \end{cases}$$

Theorem 2 holds.

Key point: scaling invariant law of α-stable law, or power law.

Proof of Lemma 1:

$\Leftarrow$ If

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{i=N}^{\infty} |X^i(t)|^2 = 0, \quad \text{with probability } 1, \tag{9}$$

then for any $t \in [0,\infty)$, for any $\epsilon > 0$, by Eq.(9), there exists $N_{t,\omega,\epsilon} \in \mathbb{N}$ satisfying $\sup_{s\in[0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^{\infty} |X^i(s)|^2 \leq \epsilon$.

$$\limsup_{s'\downarrow t} \|X(s') - X(t)\|_H^2 \tag{10}$$

$$\leq \lim_{s'\downarrow t} \sum_{i=1}^{N_{t,\omega,\epsilon}} |X^i(s') - X^i(t)|^2 + 2 \sup_{s\in[0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^{\infty} |X^i(s)|^2 \leq 2\epsilon. \tag{11}$$

$\Rightarrow$ • V is a separable Hilbert space,

$$K \text{ is a compact set in } V$$

$$\Leftrightarrow$$

$$K \text{ is bounded, closed and,}$$

and for any orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ in V, for any $\epsilon > 0$, there is a $N_\epsilon \in \mathbb{N}$

$$\sup_{x\in K} \sum_{i=N_\epsilon}^{\infty} \langle x, v_i\rangle_V^2 < \epsilon.$$

• By the Proposition 1.1 in [10], for any $x \in D([0,T], H)$,

$$\{x(t), t \in [0,T]\} \cup \{x(t-), t \in [0,T]\}$$

is a compact set in H.

[10] Jakubowski, A. *On the Skorohod topology,* Ann. Inst. Henri Poincaré 22(1986), 263-285.

4 Some Discussions

4.1. Conclusions

We give a necessary and sufficient condition of càdlàg modification of Ornstein-Uhlenbeck process with cylindrical stable noise in a Hilbert space. By using this condition, we deny a conjecture and answer some questions.

4.2. Further problems

- $X \in B$, Banach space, ?

- Itô-Stratonovich type SPDE and interacting diffusions driven by stable processes. Time (ir)regularity ? such as Parabolic Andersen Model on $\mathbb{Z}^d$.

$$dX_i(t) = \kappa \sum_{j \in \mathbb{Z}^d} a(i,j) X_j(t) dt + X_i(t-) dL_i(t), \quad i \in \mathbb{Z}^d.$$

[11] Furuoya, T., Shiga, T., *Sample lyapunov exponent for a class of linear Markovian systems over* $\mathbb{Z}^d$. Osaka J. Math 35 (1998) 35-72

THANKS