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Stochastic Equations and Lamperti transformations

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Summary

Two classes of Markov processes, continuous-state branching processes and non-negative self-similar processes, can be constructed from Lévy processes by Lamperti transformations.

A drawback of (the original form of) those transformations is that they only give representations of the processes up to their hitting times to zero.

We review some recent results on pathwise uniqueness and strong solutions of stochastic equations, which can be used to construct several natural classes Markov processes including some of those related to the Lamperti transformations.

1. A simple (bad) example

- Let $0 < \beta < 1$. Then both $x_1(t) = 0$ and $x_2(t) = ct^{1/(1-\beta)}$ solve (for a suitable $c > 0$)

$$dx(t) = x(t)^\beta dt. \tag{1}$$

Example Let $\{\tilde{N}(t)\}$ be a compensated Poisson process. Then for any $0 < \beta < 1$ one can find a β-Hölder continuous function $x \mapsto \phi(x)$ so that there are infinitely many solutions to

$$dx(t) = \phi(x(t-))d\tilde{N}(t). \tag{2}$$

In fact, between the jumps the above equation reduces to ($dt =$ compensator)

$$dx(t) = -\phi(x(t))dt.$$

- The compensated Poisson noise brings difficulties for the pathwise uniqueness.

Bass (*Probab. Surv.* 1 ('04), 1-19):

This paper is a survey of some aspects of stochastic differential equations (SDEs) with jumps. As will quickly become apparent, the theory of SDEs with jumps is nowhere near as well developed as the theory of continuous SDEs, and in fact, is still in a relatively primitive stage. In my opinion, the field is both fertile and important. To encourage readers to undertake research in this area, I have mentioned some open problems.

2. Stochastic equations of branching processes

Let $\{\xi_{n,i}\}$ and $\{\eta_n\}$ be independent families of non-negative integer-valued i.i.d. random variables. Given $Y_0 = k$, we define a (Galton-Watson) branching process with immigration by

$$Y_n = \sum_{i=1}^{Y_{n-1}} \xi_{n,i} + \eta_n, \qquad n \geq 1. \tag{3}$$

- Let $m = \mathbf{E}(\xi_{1,1})$ and write

$$Y_n - Y_{n-1} = \frac{\sqrt[\alpha]{Y_{n-1}}}{\sqrt[\alpha]{Y_{n-1}}} \sum_{i=1}^{Y_{n-1}} (\xi_{n,i} - m) + (m-1)Y_{n-1} + \eta_n. \tag{4}$$

- A suitable scaling limit leads to a typical continuous-state branching process with immigration defined by $(1 - m \rightsquigarrow b)$:

$$dy(t) = \sqrt[\alpha]{y(t-)}dz_0(t) - by(t-)dt + dz_1(t), \tag{5}$$

where $z_0(t)$ is a Brownian motion $(\alpha = 2)$ or a spectrally positive α-table process $(1 < \alpha < 2)$, and $z_1(t)$ is a subordinator. **Existence and uniqueness?**

Theorem 1 (Fu and L '10) *For a Brownian motion $B(t)$ and a spectrally positive α-table $(1 < \alpha < 2)$ process $z_0(t)$ determined by $\nu_0(dz) = cz^{-1-\alpha}dz$, there is a pathwise unique non-negative strong solution to*

$$dy(t) = \sqrt{2ay(t)}dB(t) + \sqrt[\alpha]{cy(t-)}dz_0(t) - by(t)dt \quad (+\cdots). \tag{6}$$

● The solution of (6) is a continuous-state branching process with generator A given by

$$Af(x) = x\Big[af''(x) - bf'(x) + \int_0^\infty [f(x+z) - f(x) - zf'(x)]\nu_0(dz)\Big] \; (+\cdots).$$

Theorem 2 (Dawson and L '06) *For a general Lévy measure $\nu_0(dz)$, the process can be constructed as the pathwise unique positive strong solution to*

$$\begin{aligned} y(t) \;=\; & x + \int_0^t \sqrt{2ay(s)}dB(s) - b\int_0^t y(s)ds \\ & + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{N}_0(ds,dz,du) \quad (+\cdots), \end{aligned} \tag{7}$$

where $B(t)$ is a Brownian motion and $\tilde{N}_0(ds,dz,du)$ is a compensated Poisson random measure with intensity $ds\nu_0(dz)du$.

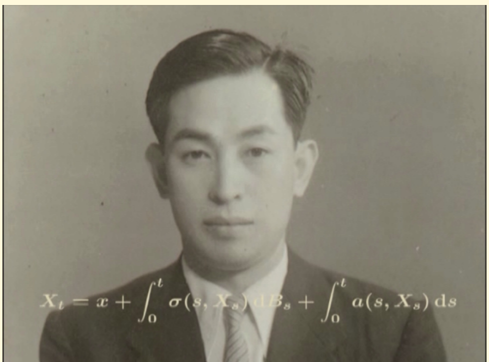
3. Brownian motion and Feller branching diffusion

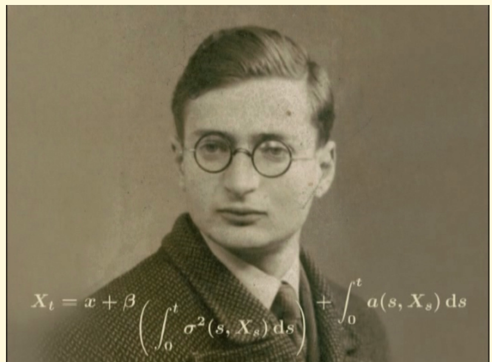
A Feller branching diffusion X_t is a special continuous-sate branching process defined by (Itô form)

$$X_t = x + \int_0^t \sqrt{X_s} dB_s, \tag{8}$$

or by (Döblin form)

$$X_t = x + B_{(\int_0^t X_s ds)}. \tag{9}$$





4. The first Lamperti transformation

Recall that a general continuous-state branching process can be defined by (Dawson and L '06):

$$\begin{aligned} y(t) \;=\; & x + \sqrt{2a}\int_0^t \sqrt{y(s)}dB(s) - b\int_0^t y(s)ds \\ & + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{\mathbf{N}}_0(ds,dz,du). \end{aligned} \tag{10}$$

It can also be defined by (Caballero, Pérez-Garmendia and Uribe-Bravo '12+):

$$y(t) = x + \sqrt{2a}B_0\Big(\int_0^t y(s)ds\Big) - b\int_0^t y(s)ds + L_0\Big(\int_0^t y(s)ds\Big), \tag{11}$$

where $\sqrt{2a}B_0(t) - bt + L_0(t) =: Z(t)$ is a spectrally positive Lévy process.

Theorem 3 (Lamperti '67; Caballero, Lambert and Uribe-Bravo '09) *Let* $z(t) = y(\kappa(t))$ *and* $\tau_0 := \inf\{t \geq 0 : z(t) = 0\}$, *where*

$$\kappa(t) = \inf\Big\{u \geq 0 : F(u) := \int_0^u y(s)ds \geq t\Big\}.$$

Then $\{z(t) : t \geq 0\}$ *is a one-sided Lévy process stopped at* τ_0.

5. Stochastic equations and measure-valued processes

● A non-decreasing family of continuous-state branching processes $\{(Y_t(x))_{t\geq 0} : x \geq 0\}$ can be defined as the unique strong solution flow of

$$\begin{aligned} Y_t(x) \;=\; & x + \sqrt{2a}\int_0^t \int_0^{Y_s(x)} W(ds,du) - b\int_0^t Y_s(x)ds \\ & + \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z\tilde{N}_0(ds,dz,du), \end{aligned} \tag{12}$$

where $W(ds,du)$ is a white noise with intensity $dsdu$ and $\tilde{N}_0(ds,dz,du)$ is a compensated Poisson random measure with intensity $ds\nu_0(dz)du$.

Theorem 4 (Dawson and L '12) *The non-deceasing function $x \mapsto Y_t(x)$ defines a random measure X_t on $[0,\infty)$ and $\{X_t : t \geq 0\}$ is a measure-valued branching process (Dawson-Watanabe process).*

Theorem 5 (Dawson and L '12) *(a similar construction for the generalized Fleming-Viot process of Bertoin and Le Gall ('03, '05, '06) using a different equation)*

6. General results on stochastic equations

Let $B(t)$ be a Brownian motion.

Consider Polish spaces U_0 and U_1. For $i = 0, 1$ let $\{N_i(ds, du)\}$ be a Poisson random measure on $(0, \infty) \times U_i$ with intensity $ds\nu_i(du)$.

Let $\tilde{N}_0(ds, du) = N_0(ds, du) - ds\nu_0(du)$ be the compensated measure.

- We study pathwise uniqueness and strong solutions of the equation:

$$x(t) = x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) + \int_0^t b(x(s))ds + \int_0^t \int_{U_1} g_1(x(s-), u)N_1(ds, du), \tag{13}$$

- Suppose that some "admissibility" conditions so that any solution to (13) is non-negative.

- **Observation** If the solution has a jump at time $s \geq 0$ given by $N_0(ds, du)$, it jumps from $x(s-)$ to $x(s) = x(s-) + g_0(x(s-), u)$ for some $u \in U_0$.

Theorem 6 (Dawson and L '12; L and Pu '12+) *There exists a pathwise unique non-negative strong solution to (13) if the following conditions are satisfied:*

(a) *there is a non-decreasing and concave function $z \mapsto r(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r(z)^{-1} dz = \infty$ and*

$$|b_1(x) - b_1(y)| + \int_{U_2} |g_1(x,u) - g_1(y,u)| \nu_1(du) \leq r(|x-y|);$$

(b) $x \mapsto x + g_0(x,u)$ *is non-decreasing for all $u \in U_0$ and there is a constant $K \geq 0$ such that*

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} |g_0(x,u) - g_0(y,u)|^2 \nu_0(du) \leq K|x-y|.$$

● Under the stronger condition that $x \mapsto g_0(x,u)$ is non-decreasing, some results on the pathwise uniqueness and strong solutions (13) were established by Fu and L ('10), L and Mytnik ('11); see also Dawson and L ('06, '12).

Theorem 7 (L and Mytnik '11; L and Pu '12+) *Suppose that the above condition (a) and the following (c) hold:*

(c) *there are constants $K \geq 0$, $p > 0$ and $0 \leq c \leq 1$ so that $x \mapsto cx + g_0(x, u)$ is non-decreasing for all $u \in U_0$ and*

$$|\sigma(x) - \sigma(y)|^2 \leq K|x - y|, \quad |g_0(x, u) - g_0(y, u)| \leq |x - y|^p f(u),$$

where $u \mapsto f(u)$ is a strictly positive function on U_0 satisfying

$$\int_{U_0} [f(u) \wedge f(u)^2] \nu_0(du) < \infty.$$

Then we have $1 \leq \alpha \leq 2$, where

$$\alpha := \inf\Big\{\beta > 1 : \limsup_{x \to 0+} x^{\beta - 1} \int_{U_0} f(u) 1_{\{f(u) \geq x\}} \nu_0(du) = 0\Big\}. \quad (14)$$

If (i) $c = 1$, $\alpha = 2$ and $p = 1/2$ or (ii) $c < 1$, $\alpha < 2$ and $1 - 1/\alpha < p \leq 1/2$, then there exists a pathwise unique strong solution to (13).

Remark In case (ii) we can have $0 < p < 1/2$, weaker than the $(1/2)$-Hölder.

7. Self-similar Markov processes

A real-valued Markov process $\{X_t : t \geq 0\}$ is self-similar with index $\alpha > 0$ if $\{c^{-\alpha} X_{ct} : t \geq 0\}$ and $\{X_t : t \geq 0\}$ have the same transition semigroup for every $c > 0$.

Example A 1-self-similar continuous-state branching process with immigration/emigration $\{x(t) : t \geq 0\}$ is defined by

$$dx(t) = 2\sqrt{x(t)}dB(t) + \delta 1_{\{x(t)>0\}}dt, \tag{15}$$

where $\delta \in \mathbb{R}$. ("$\delta > 0$" $=$ immigration; "$\delta < 0$" $=$ emigration)

Example For $1 < \alpha < 2$, a $(\alpha - 1)^{-1}$-self-similar continuous-state branching process with immigration $\{y(t) : t \geq 0\}$ is defined by

$$dy(t) = \sqrt[\alpha]{y(t)}dz_0(t) + dz_1(t), \tag{16}$$

where $\{z_0(t) : t \geq 0\}$ is a spectrally positive α-stable process and $\{z_1(t) : t \geq 0\}$ is a non-negative $(\alpha - 1)$-stable process.

8. The second Lamperti transformation

For any non-negative α-self-similar Markov process $\{X_t : t \geq 0\}$ with $X_0 = x > 0$ there is a Lévy process $\{\xi_t : t \geq 0\}$ (possibly killed with $-\infty$ as the cemetery) with $\xi_0 = 0$ such that

$$X_t = x \exp\{\xi_{\tau(x^{-1/\alpha}t)}\}, \qquad 0 \leq t \leq T_0, \tag{17}$$

where $T_0 = \inf\{t \geq 0 : X_t = 0\}$ and

$$\tau(t) = \inf\left\{u \geq 0 : I(u) := \int_0^u e^{\xi_s/a} ds \geq t\right\}. \tag{18}$$

- Two cases of the hitting time T_0:

(1) $T_0 = \infty$ (a.s.) $\Leftrightarrow$ I maps $[0, \infty)$ to $[0, \infty)$ (a.s.);

(2) $T_0 < \infty$ (a.s.) $\Leftrightarrow$ I maps $[0, \infty)$ to $[0, x^{-1/\alpha}T_0)$ (a.s.).

Problem: Continuation of the process $\{X_t : t \geq 0\}$ after time T_0; Döring and Barczy ('11+), Fitzsimmons ('06), Rivero ('05), Vuolle-Apiala ('94).

9. Stochastic equations of self-similar processes

A non-negative Markov process $\{X_t : t \geq 0\}$ is α-self-similar if and only if $\{X_t^{1/\alpha} : t \geq 0\}$ is 1-self-similar. Then it suffices to study 1-self-similar non-negative Markov processes.

● If the Lévy process $\{\xi_t : t \geq 0\}$ has generator L:

$$Lf(x) = af''(x) + bf'(x) + \int_{\mathbb{R}} \big[f(x+z) - f(x) - zf'(x)\big]\nu(dz),$$

the non-negative 1-self-similar Markov process $\{X_t : t \geq 0\}$ obtained by the second Lamperti transformation has generator A: for $x > 0$,

$$Af(x) = af''(x) + (a + b + c_\nu)f'(x) + \frac{1}{x}\int_{\mathbb{R}} \big[f(xe^z) - f(x) - zf'(x)\big]\nu(dz)$$

for some $c_\nu \geq 0$ depending on ν.

● The stochastic equation corresponding to A would provide insights into the extension(s) of $\{X_t : t \geq 0\}$ after time $T_0 := \inf\{s \geq 0 : X_s = 0\}$.

- A possible form of the stochastic equation corresponding to A could be

$$X_t = x + (a + b + c_\nu)t + \int_0^t \sqrt{2aX_s}dB_s + \int_0^t \int_{\mathbb{R}} \int_0^{\frac{1}{X_{s-}}} X_{s-}(e^z - 1)\tilde{N}(ds, dz, du). \quad (19)$$

where B_t is a Brownian motion and $\tilde{N}(ds, dz, du)$ is a compensated Poisson random measure with intensity $ds\nu(dz)du$.

Observation If the solution has a jump at time $s \geq 0$, it jumps from X_s to $X_s = X_{s-}e^z$ for some $z \in \mathbb{R}$. The jump times accumulate at $s \geq 0$ as $X_{s-} = 0$.

Theorem 8 (Döring and Barczy '11+; L and Pu '12+) *Suppose that* $\mathrm{supp}(\nu) \subset (-\infty, 0)$*. If* $(a + b + c_\nu) > 0$*, then for any* $x \geq 0$*, there is a pathwise unique non-negative strong solution* $\{X_t : t \geq 0\}$ *to (19), which is a* 1*-self-similar process.*

Open Problem How about the result when $\mathrm{supp}(\nu) \cap (0, \infty) \neq \emptyset$?

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Thanks!

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