Transition Function of a Fleming-Viot Process and a Random Time Change

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The 8th Workshop on Markov Processes and Related Topics, Beijing and Wuyi Mountain.

July 16-21, 2012
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Gamma and Dirichlet Processes

1 Definitions

Let $S$ be Polish space, $\nu_0$ a probability on $S$, $\theta$ and $\beta$ any two positive numbers.

**Definition:** The *Gamma process* with shape parameter $\theta \nu_0$ and scale parameter $\beta$ is given by

$$\mathcal{Y}^\beta_{\theta,\nu_0}(\cdot) = \beta \sum_{i=1}^{\infty} \gamma_i \delta_{\xi_i}(\cdot)$$

where $\gamma_1 > \gamma_2 > \cdots$ are the points of the inhomogeneous Poisson point process on $(0, \infty)$ with mean measure $\theta x^{-1} e^{-x} \, dx$, and independently, $\xi_1, \xi_2, \ldots$ are i.i.d. with common distribution $\nu_0$. 
Denote the law of \( \mathcal{Y}_{\theta,\nu_0}^\beta \) by \( \Gamma_{\theta,\nu_0}^\beta \). The corresponding Laplace functional has the form:

\[
\int_{M(S)} e^{-\langle \mu, g \rangle} \Gamma_{\theta,\nu_0}^\beta (d\mu) = \exp\{-\theta\langle \nu_0, \log(1 + \beta g) \rangle\}
\]

where \( M(S) \) is the set of all non-negative finite measures on \( S \), and

\[ g(s) > -1/\beta, \text{ for all } s \in S. \]
Set

\[ \sigma = \sum_{i=1}^{\infty} \gamma_i, \]

\[ P_i = \frac{\gamma_i}{\sigma}, \ i = 1, 2, \ldots, \]

\[ X_{\theta, \nu_0}(\cdot) = \sum_{i=1}^{\infty} P_i \delta_{\xi_i}(\cdot). \]

The law of \((P_1, P_2, \ldots)\) is the **Poisson-Dirichlet distribution** with parameter \(\theta\), \(X_{\theta, \nu_0}(\cdot)\) is the **Dirichlet process** with law denoted by \(\Pi_{\theta, \nu_0}\), and

\[ X_{\theta, \nu_0}(\cdot) = \frac{\Upsilon^{\beta}_{\theta, \nu_0}(\cdot)}{\Upsilon^{\beta}_{\theta, \nu_0}(S)}. \] (1)
2 Algebraic Relations

Additive property: for independent $\mathcal{Y}^\beta_{\theta_1,\nu_1}$ and $\mathcal{Y}^\beta_{\theta_2,\nu_2}$

$\mathcal{Y}^\beta_{\theta_1,\nu_1} + \mathcal{Y}^\beta_{\theta_2,\nu_2} \overset{d}{=} \mathcal{Y}^\beta_{\theta_1+\theta_2,\frac{\theta_1}{\theta_1+\theta_2}\nu_1+\frac{\theta_2}{\theta_1+\theta_2}\nu_2}$,

where $\overset{d}{=}$ denotes the equality in distribution.

Mixing: for independent $\eta \sim Beta(\theta_1, \theta_2)$, $\mathcal{X}_{\theta_1,\nu_1}$, and $\mathcal{X}_{\theta_2,\nu_2}$

$\eta \mathcal{X}_{\theta_1,\nu_1} + (1 - \eta) \mathcal{X}_{\theta_2,\nu_2} \overset{d}{=} \mathcal{X}_{\theta_1+\theta_2,\frac{\theta_1}{\theta_1+\theta_2}\nu_1+\frac{\theta_2}{\theta_1+\theta_2}\nu_2}$. 
A Measure-Valued Branching Process with Immigration

1 Definition

Let $S$ be a compact metric space, $\nu_0$ a probability on $S$, $\theta > 0$ and $\lambda > 0$. The measure-valued branching process with immigration, $Y_t$, has the following generator

$$
L = \frac{1}{2} \int_S \mu(dx) \frac{\delta^2}{\delta \mu(x)^2} + \frac{\theta}{2} \int_S \nu_0(dx) \frac{\delta}{\delta \mu(x)} - \frac{\lambda}{2} \int \mu(dx) \frac{\delta}{\delta \mu(x)}
$$

where

$$
\frac{\delta \phi}{\delta \mu(x)} = \lim_{\varepsilon \to 0} \frac{\phi(\mu + \varepsilon \delta x) - \phi(\mu)}{\varepsilon}.
$$

The process is reversible with reversible measure $\Gamma^{1/\lambda}_{\theta, \nu_0}$. 
2 Shiga's Representation

Let

\[ C(\lambda, t) = \lambda^{-1}(e^{\lambda t/2} - 1) \]
\[ C_t = 1/C(\lambda, t). \]

The entrance law for the diffusion generated by

\[ \frac{y}{2} \frac{d^2}{dy^2} - \lambda y \frac{d}{dy} \]

is given by

\[ \gamma_{\lambda, t}(dz) = e^{-\lambda t} C_t^2 \exp(-C_t z), \quad z > 0, \]

and \( \gamma_{\lambda, t}(\{0\}) = +\infty. \)
Let $\mu$ be a fixed finite measure on $S$, $W$ denote the space of all excursion pathes from $[0, \infty)$ to $[0, \infty)$, and $Q^\lambda$ be the unique $\sigma$-finite measure (excursion law) on $W$ derived from the entrance law and the transition function of the above diffusion. Consider two independent Poisson random measures $N^{\mu, \lambda}(dxdw)$ and $N_{\lambda, \theta, \nu_0}(dt dxdw)$ on spaces $S \times W$ and $[0, \infty) \times S \times W$ with respective mean measures

$$\mu(dx)Q^\lambda(dw) \text{ and } \theta dt \nu_0(dx)Q^\lambda(dw).$$

Then Shiga (1990) shows that

$$Y_t(dx) = \int_W w(t) N^{\mu, \lambda}(dxdw) + \int_{[0, \infty) \times W} w(t - s) N_{\lambda, \theta, \nu_0}(dxdw)$$

$$= \text{mass distribution of original descendent} + \text{mass distribution of immigrant}$$

A comprehensive study of this representation is found in Li (2010).
3 Transition function (Ethier and Griffiths (1993b))

Let $N_t$ be the standard Poisson process with rate one and set

$$\tilde{N}^a(t) = N_{a/C(\lambda, t)}, \ a > 0$$

$$q_{n, \lambda}^a(t) = P\{\tilde{N}(t) = n\}, n = 0, 1, 2, \ldots.$$

The process $\tilde{N}^a(t)$ is a time inhomogeneous pure death Markov chain with death rate $n/2C(-\lambda, t), n \geq 0, t > 0$ and entrance boundary $+\infty$, and $q_{n, \lambda}^a(t)$ is the corresponding marginal distribution. Let $\eta_n = \frac{1}{n} \sum_{i=1}^{n} \delta x_i$. Then the transition function of this process is given by

$$P_1(t, \mu, \cdot) = q_0^{\mu(S'), \lambda}(t) \Pi_{\theta, \nu_0}^{C(-\lambda, t)}(\cdot)$$

$$+ \sum_{n=1}^{\infty} q_n^{\mu(S), \lambda}(t) \int_{S^n} \left( \frac{\mu}{\mu(S')} \right)^n (d x_1 \times \cdots \times d x_n) \Pi_{\theta, \nu_0}^{C(-\lambda, t)}(n+\theta, \eta_{n+\theta}^{\theta+n} \eta_{n+\theta+n}^{\theta+n} \nu_0(\cdot)).$$
Let $\tilde{N}(t) = \tilde{N}^{\mu(S)}(t)$.

**Marginal distribution:** Given $Y_0 = \mu, \tilde{N}(t) = n$ and the locations of the $n$ particles, it follows that for any $t > 0$

$$Y_t \overset{d}{=} \mathcal{Y}^C_{n,\eta_n} + \mathcal{Y}^C_{\theta,\nu_0}(-\lambda, t).$$

Noting that time appears only in the scale parameter.
1 Definition

The Fleming-Viot process with parent independent mutation \( \{X_t\} \) is a probability-valued process with generator

\[
A = \frac{1}{2} \int_S \nu(dx) \left( \delta_x(dy) - \nu(dy) \right) \frac{\delta^2}{\delta \nu(y) \delta \nu(x)} + \frac{\theta}{2} \int_S \left( \nu_0(dx) - \nu(dx) \right) \frac{\delta}{\delta \nu(x)}.
\]

Reversible measure: \( \Pi_{\theta, \nu_0} \).
2 Kingman’s Coalescent

For each $n \geq 1$, let $E_n = \{1, 2, \ldots, n\}$ and $\mathcal{E}_n$ denote the collection of equivalence relations of $E_n$. Each element of $\mathcal{E}_n$ is thus a subset of $E_n \times E_n$. For example, in the case of $n = 3$, the set

$$\{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$$

defines an equivalence relation that results in two equivalence classes $\{1, 3\}$ and $\{2\}$. The set $\mathcal{E}_n$ is clearly finite and its elements will be denoted by $\eta, \xi$, etc.

In genetic applications, the equivalence relations are defined through the ancestral structures. Two individuals are equivalent if they have the same ancestor at some time $t$ in the past. For $\xi, \eta$ in $\mathcal{E}_n$, we write $\xi \prec \eta$ if $\eta$ is obtained from $\xi$ by combining exactly two equivalence classes of $\xi$ into one. For distinct $\xi, \eta$ in $\mathcal{E}_n$, set

$$q_{\xi \eta} = \begin{cases} 1, & \xi \prec \eta \\ 0, & \text{else.} \end{cases}$$
Let \(|\xi|\) be the number of equivalence classes induced by \(\xi\). Define

\[
q_\xi := -q_{\xi,\xi} = \binom{|\xi|}{2}.
\]

Definition: Kingman’s \(n\)-coalescent is a \(E_n\)-valued, continuous-time, Markov chain \(X_t\) with infinitesimal matrix \((q_{\xi,\eta})\) starting at \(X_0 = \{(i, i) : i = 1, ..., n\}\). Kingman’s coalescent corresponds to the limit process as \(n\) tends to infinity.

Let \(D_t = |X_t|\).

Then \(D_t\) is a pure-death process with death rate

\[
\lim_{h \to 0} h^{-1} P\{D_{t+h} = k - 1 \mid D_t = k\} = \binom{k}{2}, \quad k \geq 2.
\]

Replace \(\binom{k}{2}\) with \(n(n + \theta - 1)/2\) leads to Kingman’s coalescent with mutation \(D_t^\theta\).
3 Transition function (Ethier and Griffiths (93a))

Let \( \{d_\theta^n(t) : t > 0, n = 0, 1, \ldots \} \) denote the marginal distribution of Kingman’s coalescent \( D_t^\theta \), i.e.,

\[
d_\theta^n(t) = P\{D_t^\theta = n\}.
\]

Then the transition function of the Fleming-Viot process is given by

\[
P_2(t, \nu, \cdot) = d_\theta^0(t) \Pi_{\theta, \nu_0}(\cdot) + \sum_{n=1}^{\infty} d_\theta^n(t) \int_{S^n} \nu^n(dx_1 \times \cdots \times dx_n) \Pi_{n+\theta, \frac{n}{\theta+n+1} \nu_0 + \theta/n + \nu_0}(\cdot).
\]

Given \( D_t^\theta = n \) and the types of the \( n \) individuals, the marginal distribution of the process is given by

\[
X_t \overset{d}{=} \eta X_{n, \eta_n} + (1 - \eta) X_{\theta, \nu_0}, \eta \sim Beta(n, \theta).
\]
Comparison between $P_1(t, \mu, \cdot)$ and $P_2(t, \nu, \cdot)$: a termwise Gamma-Dirichlet algebra between

$$\Gamma^{C(-\lambda,t)}_{n+\theta, \frac{n}{\theta+n} \eta n + \frac{\theta}{\theta+n} \nu_0}(\cdot) \text{ and } \Pi^{n+\theta, \frac{n}{\theta+n} \eta n + \frac{\theta}{\theta+n} \nu_0}(\cdot).$$

A Problem Proposed by Ethier and Griffiths(93b): Derivation of $P_2(t, \nu, \cdot)$ from $P_1(t, \nu, \cdot)$.

This is related to the derivation of $d_{\theta}^{\mu}(t)$ from $q_{n}^{\mu(S), \lambda}(t)$ through a random time change.

Recall that $\tilde{N}(t)$ is the time inhomogeneous Markov chain $\tilde{N}_{\mu(S)/C(\lambda,t)}$. Define $\tau_t : [0, \infty) \to [0, \infty)$ by

$$t = \int_{0}^{\tau_t} \frac{du}{(\tilde{N}(u) \lor 1 + \theta - 1)C(-\lambda, u)}.$$
Theorem 1. (F and Xu (12)) The process $\tilde{N}(\tau_t)$ is the embedded chain of Kingman’s coalescent $D_t^\theta$.

As an application of this result, we get the following derivation of the fixed time distribution of the Fleming-Viot process from the measure-valued branching process with immigration.

**Theorem 2.** For any $t > 0$, let

$$\nu(\theta, t) = \frac{\tilde{N}(t)\eta\tilde{N}(t) + \theta\nu_0}{\tilde{N}(t) + \theta}.$$ 

Then we have

$$Y_t \overset{d}{=} \mathcal{Y}_C(-\lambda, t)\frac{\tilde{N}(t)}{\tilde{N}(t) + \theta, \nu(\theta, t)}$$

$$X_t \overset{d}{=} \mathcal{Y}_C(-\lambda, t)\frac{\tilde{N}(\tau_t)}{\tilde{N}(\tau_t) + \theta, \nu(\theta, \tau_t)}(S).$$
References


