Variation of optimal transport maps in Sobolev spaces

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This talk is based on a work with V. Nolot: *Sobolev estimates for optimal transport maps on Gaussian spaces*.

Problem: Let on $\mathbb{R}^d$,

$$(T_1)_* : e^{-V_1}\gamma \to e^{-W_1}\gamma,$$

$$(T_2)_* : e^{-V_2}\gamma \to e^{-W_2}\gamma$$

be optimal transport maps. How to compute the variation

$$T_1 - T_2$$

with dimension free estimates?

Facts:

Let $\nabla\Phi : e^{-V}dx \to e^{-W}dx$ be the optimal transport map.

L.V. Caffarelli (2000'): if W is convex and $V = \frac{|x|^2}{2}$, then

$$|\nabla\Phi(x) - \nabla\Phi(y)| \leq |x - y|.$$

A.V. Kolesnikov (2011'): V, W are smooth, bounded from below and $\nabla^2 W \geq K_1 Id$ with $K_1 > 0$, then Φ is smooth and

$$\sup_{x\in\mathbb{R}^d} ||\nabla^2\Phi(x)||_{HS} < +\infty.$$

But the constant on the right hand is dimension dependent.

A dimension free estimate has also been given by A.V. Kolesnikov (2011'):

$$\int_{\mathbb{R}^d} |\nabla V|^2 e^{-V}\, dx \geq K_1 \int_{\mathbb{R}^d} ||\nabla^2\Phi||_{HS}^2\, e^{-V} dx.$$

Although the constant K_1 is of dimension free, but on infinite dimensional spaces, $\nabla^2\Phi$ usually is not of Hilbert-Schmidt class. Let $\nabla\Phi(x) = x + \nabla\varphi(x)$. A dimension free inequality for $||\nabla^2\varphi||_{HS}^2$ has been established by A.V. Kolesnikov (2011'), under the hypothesis

$$\nabla^2 W \leq K_2\, Id.$$

Related works have been done by G. De Philippis and A. Figalli. See arXive 2011 and 2012.

Here are our results:

Theorem (A)

Let $e^{-V}d\gamma$ and $e^{-W}d\gamma$ be two probability measures on $\mathbb{R}^d$, where γ is the standard Gaussian measure on $\mathbb{R}^d$. Suppose that $\nabla^2 W \geq -c\, Id$ with $c \in [0,1[$. Then

$$\begin{aligned} &\int_{\mathbb{R}^d} |\nabla V|^2 e^{-V} d\gamma - \int_{\mathbb{R}^d} |\nabla W|^2 e^{-W} d\gamma \\ &\quad + \frac{2}{1-c} \int_{\mathbb{R}^d} ||\nabla^2 W||^2_{HS}\, e^{-W} d\gamma \\ &\geq 2\mathrm{Ent}_\gamma(e^{-V}) - 2\mathrm{Ent}_\gamma(e^{-W}) \\ &\quad + \frac{1-c}{2} \int_{\mathbb{R}^d} ||\nabla^2 \varphi||^2_{HS}\, e^{-V} d\gamma. \end{aligned}$$

Theorem (B)

Let $\nabla\Phi_1 : e^{-V_1}\gamma \to e^{-W_1}\gamma$ *and* $\nabla\Phi_2 : e^{-V_2}\gamma \to e^{-W_2}\gamma$*. Assume that* $\nabla^2 W_1 \geq -c\, Id$ *for* $c < 1$*, then*

$$\int_{\mathbb{R}^d} |\nabla\Phi_1 - \nabla\Phi_2|^2 \, e^{-V_2} d\gamma$$

$$\leq \frac{4}{1-c}\mathrm{Ent}_{e^{-V_1}\gamma}(e^{V_1 - V_2}) + \frac{4}{(1-c)^2}\int |\nabla(W_1 - W_2)|^2 e^{-W_2} d\gamma.$$

Theorem (C)

Let $1 \leq p < 2$ *and*

$$M(\varphi_1, \varphi_2) = \Big\| ||I + \nabla^2 \varphi_1||_{op} \Big\|^2_{L^{\frac{2p}{2-p}}(e^{-V_2}\gamma)} \vee \Big\| ||I + \nabla^2 \varphi_2||_{op} \Big\|^2_{L^{\frac{2p}{2-p}}(e^{-V_2}\gamma)}.$$

Assume that $\nabla^2 W_1 \geq -c\, Id$ *with* $c \in [0,1[$*. Then we have*

$$\begin{aligned} ||\nabla^2 \varphi_1 - \nabla^2 \varphi_2||^2_{L^p(e^{-V_2}\gamma)} \leq & 2M(\varphi_1, \varphi_2) \Big[3 \int_{\mathbb{R}^d} (V_1 - V_2) e^{-V_2} d\gamma \\ & + \frac{2}{1-c} \int_{\mathbb{R}^d} |\nabla (W_1 - W_2)|^2 e^{-W_2} d\gamma \Big]. \end{aligned}$$

Application:

Theorem (D)

Let (X, H, μ) be an abstract Wiener space. Consider $V \in \mathbb{D}_1^2(X), W \in \mathbb{D}_2^2(X)$ such that $\int_X e^{-V} d\mu = \int_X e^{-W} d\mu = 1$. Assume that for $c \in [0, 1[$,

$$0 < \delta_1 \leq e^{-V} \leq \delta_2,\ \ e^{-W} \leq \delta_2,\ \ \nabla^2 W \geq -c\, Id,$$

then there exists a function $\varphi \in \mathbb{D}_2^2(X)$ such that $x \to x + \nabla\varphi(x)$ pushes $e^{-V}\mu$ to $e^{-W}\mu$ and solves the Monge-Ampère equation

$$e^{-V} = e^{-W(T)} e^{\mathcal{L}\varphi - \frac{1}{2}|\nabla\varphi|^2}\, det_2(Id_{H\otimes H} + \nabla^2\varphi),$$

where $T(x) = x + \nabla\varphi(x)$.

Remark:

- Due to the singularity of the cost function $(x, y) \to |x - y|_H$ on $X \times X$, we do not know if $x \to T(x) := x + \nabla\varphi(x)$ in Theorem D is the optimal transport map; however if W satisfies

$$(*) \qquad W_n \in \mathbb{D}_2^2(H_n) \text{ for all } n \geq 1,$$

where $\pi_n : X \to H_n$ is a finite dimensional approximation, and $e^{-W_n} = \mathbb{E}^{\pi_n}(e^{-W})$; then T is the optimal transport map.
- The result in Theorem D includes two special cases: (i) the source measure is the Wiener measure, by Feyel and Üstünel; (ii) the target measure is the Wiener measure, by Bogachev and Kolesnikov.
- For an orthonormal basis $\{e_n; n \geq 1\}$ of H, define $W(x) = \sum_{n\geq 1} \lambda_n e_n(x)^2$, where $\lambda_n > -1/2$ and $\sum_{n\geq 1} |\lambda_n| < +\infty$; then W satisfies $(*)$.

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Proof of results:

A priori estimate:
Let $(\nabla\Phi_1)_* : h_1\gamma \to f_1\gamma$, $(\nabla\Phi_2)_* : h_2\gamma \to f_2\gamma$. Then by change of variables, for $\alpha = \frac{|x|^2}{2}$,

$$f_1(\nabla\Phi_1)e^{-\alpha(\nabla\Phi_1)}\det(\nabla^2\Phi_1) = h_1 e^{-\alpha},$$

$$f_2(\nabla\Phi_2)e^{-\alpha(\nabla\Phi_2)}\det(\nabla^2\Phi_2) = h_2 e^{-\alpha}.$$

Let S_2 be the inverse map of $\nabla\Phi_2$. Acting on the right by S_2, we get

$$f_1(\nabla\Phi_1(S_2))e^{-\alpha(\nabla\Phi_1(S_2))}\det(\nabla^2\Phi_1(S_2)) = h_1(S_2)e^{-\alpha(S_2)},$$

$$f_2\, e^{-\alpha}\ \det(\nabla^2\Phi_2(S_2)) = h_2(S_2)e^{-\alpha(S_2)}.$$

It follows that

$$\frac{f_1}{f_2}\cdot\frac{f_1(\nabla\Phi_1(S_2))e^{-\alpha(\nabla\Phi_1(S_2))}}{f_1e^{-\alpha}}\cdot\det\left[(\nabla^2\Phi_1)(\nabla^2\Phi_2)^{-1}\right](S_2)=\frac{h_1(S_2)}{h_2(S_2)}.$$

Taking the logarithm yields

$$\log(\frac{f_1}{f_2})+\log(f_1e^{-\alpha})(\nabla\Phi_1(S_2))-\log(f_1e^{-\alpha})$$
$$+\log\det\left[(\nabla^2\Phi_1)(\nabla^2\Phi_2)^{-1}\right](S_2)=\log(\frac{h_1}{h_2})(S_2).$$

$$\begin{aligned}&\int_{\mathbb{R}^d}\log(\frac{h_1}{h_2})(S_2)\,f_2d\mu-\int_{\mathbb{R}^d}\log(\frac{f_1}{f_2})\,f_2d\gamma\\&=\int_{\mathbb{R}^d}\log\det\left[(\nabla^2\Phi_1)(\nabla^2\Phi_2)^{-1}\right](S_2)\,f_2d\gamma\qquad(1)\\&+\int_{\mathbb{R}^d}\left[\log(f_1e^{-\alpha})(\nabla\Phi_1(S_2))-\log(f_1e^{-\alpha})\right]f_2d\gamma.\end{aligned}$$

By Taylor formula up to order 2,

$$
\begin{aligned}
&\log(f_1 e^{-\alpha})(\nabla\Phi_1(S_2)) - \log(f_1 e^{-\alpha}) \\
&= \langle \nabla \log(f_1 e^{-\alpha}), \nabla\Phi_1(S_2(x)) - x\rangle \\
&+ \int_0^1 (1-t)\Big[\nabla^2 \log(f_1 e^{-\alpha})((1-t)x + t\nabla\Phi_1(S_2(x))\Big] \\
&\qquad\qquad\qquad \cdot (\nabla\Phi_1(S_2(x)) - x)^2\, dt.
\end{aligned}
$$

By integration by parts,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \langle \nabla \log(f_1 e^{-\alpha}), \nabla\Phi_1(S_2(x)) - x\rangle\, f_2 d\gamma \\
&= -\int_{\mathbb{R}^d} \mathrm{Trace}\Big[\nabla^2\Phi_1(S_2)\cdot(\nabla^2\Phi_2)^{-1}(S_2) - Id\Big]\, f_2 d\gamma \\
&\quad - \int_{\mathbb{R}^d} \langle \nabla\Phi_1(S_2(x)) - x, \nabla(\log\frac{f_2}{f_1})\rangle\, f_2 d\gamma.
\end{aligned}
$$

For a matrix A, the Fredholm-Carleman determinant $\det_2(A)$ is defined by $\det_2(A) = e^{\mathrm{Trace}(Id-A)} \det(A)$. So

$$\log \det_2(A) = \mathrm{Trace}(Id - A) + \log \det(A).$$

Theorem (B')

$$\mathrm{Ent}_{h_1\gamma}\Big(\frac{h_2}{h_1}\Big) - \mathrm{Ent}_{f_1\gamma}\Big(\frac{f_2}{f_1}\Big) = \int_{\mathbb{R}^d} \langle \nabla\Phi_1 - \nabla\Phi_2, \nabla(\log \frac{f_2}{f_1})(\nabla\Phi_2)\rangle \, h_2 d\gamma$$
$$- \int_{\mathbb{R}^d} \log \det_2\Big((\nabla^2\Phi_2)^{-1/2}\, \nabla^2\Phi_1\, (\nabla^2\Phi_2)^{-1/2}\Big)\, h_2 d\gamma$$
$$+ \int_0^1 (1-t)dt \int_{\mathbb{R}^d} \Big[-\nabla^2 \log(f_1 e^{-\alpha})((1-t)\nabla\Phi_2 + t\nabla\Phi_1)\Big] \cdot (\nabla\Phi_1 - \nabla\Phi_2)^2\, h_2 d\gamma.$$

Assume $\nabla^2(-\log f) \geq -c\, Id$; then

$$\int_{\mathbb{R}^d} |\nabla\Phi_1 - \nabla\Phi_2|^2\, h_2 d\gamma \leq \frac{4}{1-c}\Big(\mathrm{Ent}_{h_1\gamma}(\frac{h_2}{h_1}) - \mathrm{Ent}_{f_1\gamma}(\frac{f_2}{f_1})\Big) + \frac{4}{(1-c)^2}\int_{\mathbb{R}^d} |\nabla \log \frac{f_2}{f_1}|^2\, f_2 d\gamma.$$

If moreover $f_1 = f_2$, then

$$\frac{1-c}{2}\int_{\mathbb{R}^d} |\nabla\Phi_1 - \nabla\Phi_2|^2\, h_2 d\gamma \leq \mathrm{Ent}_{h_1\gamma}(\frac{h_2}{h_1}).$$

It is the content of Theorem B. We will give a sketch of the proof to Theorem A. Let e^{-V} and e^{-W} be two density functions with respect to γ. Let $(\nabla\Phi)_* : e^{-V}\gamma \to e^{-W}\gamma$, that is, For $F \in C_b(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} F(\nabla\Phi)\, e^{-V} d\gamma = \int_{\mathbb{R}^d} F\, e^{-W} d\gamma.$$

Let $a \in \mathbb{R}^d$; then

$$\int_{\mathbb{R}^d} F(\nabla\Phi(x+a))e^{-V(x+a)}e^{-\langle x,a\rangle - \frac{1}{2}|a|^2}\, d\gamma = \int_{\mathbb{R}^d} F(\nabla\Phi)e^{-V}\, d\gamma.$$

Denote by τ_a the translation by a, and $M_a(x) = e^{-\langle x,a\rangle - \frac{1}{2}|a|^2}$, then

$$\nabla(\tau_a\Phi)_* : e^{-\tau_a V} M_a\gamma \to e^{-W}\gamma.$$

Let $h_1 = e^{-\tau_a V} M_a, h_2 = e^{-V}$. By Theorem B' and $\mathrm{Ent}_{h_1\gamma}(\frac{h_2}{h_1}) = \int_{\mathbb{R}^d}(\tau_a V - V + \langle x,a\rangle + \frac{1}{2}|a|^2)e^{-V} d\gamma$,then

$$\begin{aligned}
&\int_{\mathbb{R}^d}(\tau_a V - V + \langle x,a\rangle + \frac{1}{2}|a|^2)e^{-V} d\gamma \\
&\geq \int_0 1(1-t)dt \int_{\mathbb{R}^d}\Big[(Id + \nabla^2 W)(\Lambda(t,x,a))\Big] \\
&\qquad\qquad \cdot (\nabla\Phi(x) - \nabla\Phi(x+a))^2 e^{-V} d\gamma,
\end{aligned}$$

where $\Lambda(t,x,a) = (1-t)\nabla\Phi(x) + t\nabla\Phi(x+a)$. Note that as $a \to 0$, $\Lambda(t,x,a) \to \nabla\Phi(x)$. Replacing a by $-a$, and summing these inequalities, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(V(x+a) + V(x-a) - 2V(x) + |a|^2\right) e^{-V} d\gamma \\
&\geq \int_0^1 (1-t)dt \int_{\mathbb{R}^d} \Big[(Id + \nabla^2 W)(\Lambda(t,x,a))\Big] \\
&\qquad\qquad \cdot (\nabla\Phi(x) - \nabla\Phi(x+a))^2 e^{-V} d\gamma \\
&+ \int_0^1 (1-t)dt \int_{\mathbb{R}^d} \Big[(Id + \nabla^2 W)(\Lambda(t,x,-a))\Big] \\
&\qquad\qquad \cdot (\nabla\Phi(x) - \nabla\Phi(x-a))^2 e^{-V} d\gamma,
\end{aligned}$$

Replacing a by εa and dividing by ε^2 the two hand sides, letting $\varepsilon \to 0$ yields

$$\int_{\mathbb{R}^d} \left[D_a^2 V + |a|^2 \right] e^{-V} d\gamma \geq \int_{\mathbb{R}^d} (Id + \nabla^2 W)_{\nabla\Phi} \left(D_a \nabla\Phi, D_a \nabla\Phi \right) e^{-V} d\gamma.$$

By integration by parts,

$$\int_{\mathbb{R}^d} D_a^2 V \, e^{-V} d\gamma = \int_{\mathbb{R}^d} (D_a V)^2 e^{-V} d\gamma + \int_{\mathbb{R}^d} D_a V \, \langle a, x \rangle \, e^{-V} d\gamma.$$

Write $\nabla\Phi(x) = x + \nabla\varphi(x)$, then
$|D_a \nabla\Phi|^2 = |a|^2 + 2\langle a, D_a \nabla\varphi \rangle + |D_a \nabla\varphi|^2$. Summing a on an orthonormal basis gives

$$\begin{aligned} &\int_{\mathbb{R}^d} |\nabla V|^2 e^{-V} d\gamma + \int_{\mathbb{R}^d} \langle x, \nabla V \rangle \, e^{-V} d\gamma \\ &\geq 2 \int_{\mathbb{R}^d} \Delta\varphi \, e^{-V} d\gamma + \int_{\mathbb{R}^d} ||\nabla^2 \varphi||_{HS}^2 e^{-V} d\gamma \\ &\quad + \sum_{a \in \mathcal{B}} \int_{\mathbb{R}^d} \nabla^2 W_{\nabla\Phi} (D_a \nabla\Phi, D_a \nabla\Phi) \, e^{-V} d\gamma. \end{aligned}$$

Let $N_W(\nabla^2\varphi) = \sum_{a\in\mathcal{B}} \nabla^2 W_{\nabla\Phi}(D_a\nabla\varphi, D_a\nabla\varphi)$. Then

$$\begin{aligned}
&\sum_{a\in\mathcal{B}} \int_{\mathbb{R}^d} \nabla^2 W_{\nabla\Phi}(D_a\nabla\Phi, D_a\nabla\Phi)\, e^{-V} d\gamma \\
&\geq \int_{\mathbb{R}^d} (\Delta W)_{\nabla\Phi}\, e^{-V} d\gamma + 2\int_{\mathbb{R}^d} \langle \nabla^2 W_{\nabla\Phi}, \nabla^2\varphi\rangle_{HS}\, e^{-V} d\gamma \\
&\qquad - c\int_{\mathbb{R}^d} ||\nabla^2\varphi||^2_{HS}\, e^{-V} d\gamma.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla V|^2 e^{-V} d\gamma + \int_{\mathbb{R}^d} \langle x, \nabla V\rangle\, e^{-V} d\gamma \\
&\geq 2\int_{\mathbb{R}^d} \Delta\varphi\, e^{-V} d\gamma + (1-c)\int_{\mathbb{R}^d} ||\nabla^2\varphi||^2_{HS} e^{-V} d\gamma \qquad (2) \\
&+ \int_{\mathbb{R}^d} \Delta W_{\nabla\Phi}\, e^{-V} d\gamma + 2\int_{\mathbb{R}^d} \langle \nabla^2 W_{\nabla\Phi}, \nabla^2\varphi\rangle_{HS}\, e^{-V} d\gamma.
\end{aligned}$$

In order to obtain desired terms, we first use the relation

$$\int_{\mathbb{R}^d} |x + \nabla\varphi(x)|^2 \, e^{-V} d\gamma = \int_{\mathbb{R}^d} |x|^2 \, e^{-W} d\gamma$$

which gives that

$$2\int_{\mathbb{R}^d} \langle x, \nabla\varphi(x)\rangle \, e^{-V} d\gamma = \int_{\mathbb{R}^d} |x|^2 \, e^{-W} d\gamma - \int_{\mathbb{R}^d} |x|^2 \, e^{-V} d\gamma \\ - \int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 \, e^{-V} d\gamma.$$

Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator:
$\mathcal{L}f(x) = \Delta f(x) - \langle x, \nabla f\rangle$. Remark that $\mathcal{L}(\frac{1}{2}|x|^2) = d - |x|^2$. Then

$$\int_{\mathbb{R}^d} |x|^2 \, e^{-W} d\gamma - \int_{\mathbb{R}^d} |x|^2 \, e^{-V} d\gamma \\ = -\int_{\mathbb{R}^d} \langle x, \nabla W\rangle \, e^{-W} d\gamma + \int_{\mathbb{R}^d} \langle x, \nabla V\rangle \, e^{-V} d\gamma.$$

Therefore

$$2\int_{\mathbb{R}^d} \langle x, \nabla\varphi(x)\rangle\, e^{-V} d\gamma = -\int_{\mathbb{R}^d} \langle x, \nabla W\rangle\, e^{-W} d\gamma$$
$$+ \int_{\mathbb{R}^d} \langle x, \nabla V\rangle\, e^{-V} d\gamma - \int_{\mathbb{R}^d} |\nabla\varphi|^2\, e^{-V} d\gamma.$$

On the other hand, from Monge-Ampère equation,

$$e^{-V} = e^{-W(\nabla\Phi)} e^{\mathcal{L}\varphi - \frac{1}{2}|\nabla\varphi|^2} \det_2(Id + \nabla^2\varphi),$$

we have

$$\mathcal{L}\varphi = -V + W(\nabla\Phi) + \frac{1}{2}|\nabla\varphi|^2 - \log\det_2(Id + \nabla^2\varphi),$$

and

$$\int_{\mathbb{R}^d} \mathcal{L}\varphi\, e^{-V} d\gamma = \mathrm{Ent}_\gamma(e^{-V}) - \mathrm{Ent}_\gamma(e^{-W}) + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\varphi|^2\, e^{-V} d\gamma$$
$$- \int_{\mathbb{R}^d} \log\det_2(Id + \nabla^2\varphi)\, e^{-V} d\gamma.$$

Combing two above equalities gives

$$
\begin{aligned}
&2\int_{\mathbb{R}^d} \Delta\varphi\, e^{-V} d\gamma = 2\int_{\mathbb{R}^d} \mathcal{L}\varphi\, e^{-V} d\gamma + 2\int_{\mathbb{R}^d} \langle x, \nabla\varphi\rangle\, e^{-V} d\gamma \\
&= 2\mathrm{Ent}_\gamma(e^{-V}) - 2\mathrm{Ent}_\gamma(e^{-W}) - 2\int_{\mathbb{R}^d} \log \det_2 (Id + \nabla^2\varphi)\, e^{-V} d\gamma \\
&- \int_{\mathbb{R}^d} \langle x, \nabla W\rangle\, e^{-W} d\gamma + \int_{\mathbb{R}^d} \langle x, \nabla V\rangle\, e^{-V} d\gamma.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla V|^2 e^{-V} d\gamma + \int_{\mathbb{R}^d} \langle x, \nabla V\rangle\, e^{-V} d\gamma \\
&\geq 2\int_{\mathbb{R}^d} \Delta\varphi\, e^{-V} d\gamma + (1-c)\int_{\mathbb{R}^d} ||\nabla^2\varphi||_{HS}^2 e^{-V} d\gamma \qquad (2) \\
&+ \int_{\mathbb{R}^d} \Delta W\, e^{-W} d\gamma + 2\int_{\mathbb{R}^d} \langle \nabla^2 W_{\nabla\Phi}, \nabla^2\varphi\rangle_{HS}\, e^{-V} d\gamma.
\end{aligned}
$$

So we get

$$\int_{\mathbb{R}^d} |\nabla V|^2 \, e^{-V} d\gamma \geq 2\mathrm{Ent}_\gamma(e^{-V}) - 2\mathrm{Ent}_\gamma(e^{-W})$$
$$+ (1-c) \int_{\mathbb{R}^d} ||\nabla^2 \varphi||_{HS}^2 \, e^{-V} d\gamma$$
$$+ \int_{\mathbb{R}^d} \mathcal{L}W \, e^{-W} d\gamma + 2 \int_{\mathbb{R}^d} \langle \nabla^2 W(\nabla \Phi), \nabla^2 \varphi \rangle_{HS} \, e^{-V} d\gamma.$$

Now by

$$2 \int_{\mathbb{R}^d} |\langle \nabla^2 W(\nabla \Phi), \nabla^2 \varphi \rangle_{HS}| \, e^{-V} d\gamma$$
$$\leq \frac{1-c}{2} \int_{\mathbb{R}^d} ||\nabla^2 \varphi||_{HS}^2 \, e^{-V} d\gamma + \frac{2}{1-c} \int_{\mathbb{R}^d} ||\nabla^2 W||_{HS}^2 \, e^{-W} d\gamma.$$

We get Theorem A.

$$\int_{\mathbb{R}^d} |\nabla V|^2 \, e^{-V} d\gamma - \int_{\mathbb{R}^d} |\nabla W|^2 \, e^{-W} d\gamma + \frac{2}{1-c} \int_{\mathbb{R}^d} ||\nabla^2 W||_{HS}^2 e^{-W} d\gamma$$
$$\geq 2\mathrm{Ent}_\gamma(e^{-V}) - 2\mathrm{Ent}_\gamma(e^{-W}) + \frac{1-c}{2} \int_{\mathbb{R}^d} ||\nabla^2 \varphi||_{HS}^2 \, e^{-V} d\gamma.$$

Thanks a lot for your attention!