L^p-Derivatives of Jump Processes and Gradient Estimates of Their Transition Semigroups

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The 8th Workshop on Markov Processes and Related Topics
Beijing and Fujian Normal University

July 19, 2012

Outline

Consider SDEs

$$\begin{cases} dX_t = b(X_t)dt + dL_t \\ X_0 = x, \end{cases}$$

and

$$\begin{cases} dX_t = b(X_{t-})dt + \int_{\mathbb{R}_0^n} f(X_{t-}, z)\widetilde{N}(dz, dt), \\ X_0 = x. \end{cases}$$

- $b : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}_0^n \to \mathbb{R}^n$ are measurable;
- L_t is a pure jump Lévy process with Lévy measure ν, and $\widetilde{N}(dz, ds)$ is the associated martingale measure.

We aim to obtain gradient estimates, strongly Feller property and Harnack inequality for the associated semigroup

$$P_t g(x) := Eg(X_t^x) = \int_{\mathbb{R}^n} g(y) P_t(x, dy),\ 0 \le t \le T,\ g \in \mathscr{B}_b(\mathbb{R}^n).$$

Some known results:

- [R. F. Bass and M. Cranston (1986) Ann. Prob]

An integration by parts formula for pure jump processes whose characteristic measure is Lebesgue measure.

- [Norris (1988)]

An integration by parts formula for SDEs driven by jump processes with locally regular Lévy measure.

- [F. Y. Wang(2011)SPA]

Explicit gradient estimates for time-homogenous linear SDEs driven by general Lévy processes.

- [F. Y. Wang (2012)arXiv:1104.5531v4]

Derivative formula, Harnack inequality, Log-Harnack inequality for time-nonhomogenous linear SDEs driven by general Lévy processes.

- $\cdots\cdots\cdots\cdots$

Basic Assumption:

- There exists an open set $U \subset \mathbb{R}_0^n$ and a function $\rho \in C^1(U)$ with

$$\nu(dz)\mid_U = \rho(z)dz \quad and \quad \rho(z) > 0, \quad \forall z \in U.$$

- U_0 is a closed subset of U.
- A predictable process $V = \{V(t,z)\}_{t \le T}$ on $\left(\Omega \times [0,T] \times \mathbb{R}_0^n\right)$ satisfies:

(H2.1) $suppV \subseteq [0,T] \times U_0, \quad V(t,\cdot) \in C^1(U_0)$; V and $D_z V$ are bounded.

L^p-Derivatives and Integration by Parts Formula

Define a perturbed random measure,

$$N^{\epsilon}(B \times [0,t]) = \int_0^t \int_{\mathbb{R}_0^n} I_B(z + \epsilon V(s,z)) N(dz, ds), \qquad (1)$$

where $\nu(B) < \infty$.

Definition 2.1[Bass]

For $p \geq 1$, a function $F = F(N)$ is called to have an L^p-derivative in the direction V, if there exists an L^p-integrable random variable denoted by $D_V F$, such that

$$\lim_{\epsilon \to 0} E\left|\frac{F(N^{\epsilon}) - F(N)}{\epsilon} - D_V F\right|^p = 0,$$

where N^{ϵ} is defined as (1).

For $\epsilon > 0$ sufficiently small, let

$$\lambda^\epsilon(s,z) = \begin{cases} detD_z\big(z+\epsilon V(s,z)\big)\frac{\rho(z+\epsilon V(s,z))}{\rho(z)}, & z \in U_0, \\ 1, & z \in (U_0)^c, \end{cases}$$

$$Z_t^\epsilon = \exp\Big\{\int_0^t \int_U \log\lambda^\epsilon(s,z)N(dz,ds) - \int_0^t\int_U (\lambda^\epsilon(s,z)-1)\nu(dz)ds\Big\}.$$

There exists a new probability measure P^ϵ such that

$$\frac{dP^\epsilon}{dP}\,|_{\mathcal{F}_t} = Z_t^\epsilon.$$

L^p-Derivatives and Integration by Parts Formula

Lemma 2.1

$\left\{ \frac{Z_t^\epsilon - 1}{\epsilon} \right\}_{0<\epsilon<1}$ are uniformly integrable.

Proposition 2.2[Norris]

The P^ϵ-law of N_t^ϵ is qual to the P-law of N_t.

Proposition 2.3

Assume F is a function of $N_t := \{N(dz, ds), s \leq t\}$. If $F(N_t)$ is square-integrable and V satisfies (H2.1), then $F(N_t)$ has a L^1-derivative $D_V F(N_t)$ and

$$ED_V F(N_t) = -E[F(N_t)R_t], \tag{2}$$

where $R_t = \int_0^t \int_{U_0} \frac{div(\rho(z)V(s,z))}{\rho(z)} \widetilde{N}(dz, ds)$.

For an $\mathbb{R}^n$-valued Lévy process $L = \{L_t\}_{t \le T}$, with the Lévy-Khinchin decomposition,

$$L_t = \alpha t + W_t + \int_0^t \int_{[z:0<|z|\le 1]} z\widetilde{N}(dz, ds) + \int_0^t \int_{[z:|z|>1]} zN(dz, ds).$$

Set

$$\mathcal{F}_t = \sigma\{W_s : s \le t\}, \qquad \mathcal{G}_t = \sigma\{N_s : s \le t\}.$$

Then filtration $\{\mathcal{F}_t\}_{t>0}$ and $\{\mathcal{G}_t\}_{t>0}$ are independent.
Denote

$$\mathscr{N} = \Big\{ h : h \textit{ is adapted to } \{\mathcal{F}_t\}_{t \le T} \textit{ and } E \exp\{\frac{1}{2} \int_0^T |h(s)|^2 ds\} < \infty \Big\}.$$

For fixed $h \in \mathscr{N}$, $\mathcal{G}_t$-adapted process V satisfying (H2.1), let

$$
\begin{aligned}
Z_t^\epsilon = \exp\Bigg\{ & \epsilon \int_0^t \langle h(s), dW_s \rangle - \frac{\epsilon^2}{2} \int_0^t |h(s)|^2 ds \\
& + \int_0^t \int_U \log \lambda^\epsilon(s,z) N(dz,ds) - \int_0^t \int_U (\lambda^\epsilon(s,z) - 1)\nu(dz)ds \Bigg\}.
\end{aligned}
$$

There exists a new probability measure Q^ϵ such that $\frac{dQ^\epsilon}{dP} |_{\mathcal{F}_t \vee \mathcal{G}_t} = Z_t^\epsilon$, where $\mathcal{F}_t \vee \mathcal{G}_t = \sigma\{A_1 \cap A_2 | A_1 \in \mathcal{F}_t, A_2 \in \mathcal{G}_t\}$.

Definition 2.2

A function $F = F(W_t, N_t)$ is called to have an L^p-derivative in the direction (h, V), denoted by $D_{(h,V)}F$. If $D_{(h,V)}F$ is L^p-integrable and

$$\lim_{\epsilon\to 0} E\left|\frac{F\left(W_t-\epsilon\int_0^t h(s)ds, N_t^\epsilon\right)-F\left(W_t, N_t\right)}{\epsilon} - D_{(h,V)}F(W_t, N_t)\right|^p = 0,$$

where N^ϵ is defined as (1).

L^p-Derivatives and Integration by Parts Formula

Proposition 2.4

For $h \in \mathscr{N}$, $\mathcal{G}_t$-adapted process V satisfying (H2.1), if $F = F(W_t, N_t)$ are square-integrable, then F has an L^1-derivative $D_{(h,V)}F(W_t, N_t)$, and

$$ED_{(h,V)}F(W_t, N_t) = -E[F(W_t, N_t)(H_t + R_t)], \tag{3}$$

where

$$H_t = \int_0^t \langle h(s), dW_s \rangle, \qquad R_t = \int_0^t \int_{U_0} \frac{div(\rho(z)V(s,z))}{\rho(z)} \widetilde{N}(dz, ds).$$

Remark 2.5

If $h \equiv 0$, then equality (3) becomes to (2). If $V \equiv 0$, equality (3) becomes to the classic integration by parts formula associated to Wiener processes.

Outline

Consider the following equation

$$\begin{cases} dX_t = b(X_t)dt + dL_t \\ X_0 = x, \end{cases} \tag{4}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$, $L_t = \int_0^t \int_{|z|>1} zN(dz, ds) + \int_0^t \int_{0<|z|\leq 1} z\widetilde{N}(dz, ds)$.
If b satisfies Lipschitz condition, then the equation (4) admits a unique solution, written as

$$X_t = x + \int_0^t b(X_s)ds + L_t.$$

- Question: As a function of N, which direction ensures the existence of the L^2-derivative for the solution X_t?

- Hypothesis (H3.1):

$$|V(s,z)| \leq k(s)h(z), \quad \forall s \in [0,T], \quad \forall z \in \mathbb{R}_0^n,$$

where $k : [0,T] \to [0,\infty)$ is a bounded measurable function, $h : \mathbb{R}_0^n \to [0,\infty)$ satisfies $h \in L^1(U,\nu(dz)) \cap L^2(U,\nu(dz))$ and $supp\ h \subseteq U$.

Theorem 3.1

If (H3.1) holds and $b(x) \in C_b^2(\mathbb{R}^n)$, then the solution of equation(4) has an L^2-derivative. Moreover, the L^2-derivative satisfies the following equation

$$\begin{cases} dD_V X_t = \nabla b(X_t) D_V X_t dt + \int_{\mathbb{R}_0^n} V(t,z) N(dz,dt) \\ D_V X_0 = 0. \end{cases} \tag{5}$$

(H3.2)There exist a square-integrable nonnegative process $\{\varphi(s)\}_{s\leq T}$ and a nonnegative function $\psi : U \to \mathbb{R}$ satisfying $\psi \in L^2(U, \nu(dz))$ such that

$$|\frac{div(\rho(z)V(s,z))}{\rho(z)}| \leq \varphi(s)\psi(z).$$

Theorem 3.2

If (H3.1) and (H3.2) hold, $\frac{h}{dist(\cdot,\partial U)} \in L^2(U, \nu(dz))$, then for any bounded test function g,

$$ED_V g(X_t) = -Eg(X_t)R_t,$$

where

$$R_t = \int_0^t \int_U \frac{div(\rho(z)V(s,z))}{\rho(z)} \widetilde{N}(dz, ds).$$

Set

$$\theta = \liminf_{\epsilon \to 0} \frac{\ln \epsilon}{\nu(h \geq \epsilon)}$$

$$\mathcal{H}_\rho = \Big\{ h \in L^1_+(U, \nu(dz)) \cap L^2(U, \nu(dz)) \Big| \frac{|\nabla(h\rho)|}{\rho} \in L^2(U, \nu(dz)) \text{ and } \frac{h}{dist(\cdot, \partial U)} \in L^2(U, \nu(dz)) \Big\}.$$

Theorem 3.3

If $b(x) \in C_b^2(\mathbb{R}^n)$, $h \in \mathcal{H}_\rho$ and $\theta > -\infty$, then for $\xi \in \mathbb{R}^n$, $t > -4\theta$ and bounded test function g,

$$\begin{aligned} &|\nabla_\xi P_t g(x)| \\ \leq &\|g\|_\infty C(t,h,\nu) e^{\|\nabla b\|_\infty t} \|\xi\| \Bigg\{ \Big(t \int_U \frac{|\nabla(h(z)\rho(z))|^2}{\rho^2(z)} \nu(dz)\Big)^{\frac{1}{2}} \\ &\qquad + 2\sqrt{t \int_U h(z)^2 \nu(dz) + (t \int_U h(z)\nu(dz))^2} \Bigg\}, \end{aligned}$$

where $C(t,h,\nu)$ is a constant depending on t, h and ν.

Corollary 3.4

Under the conditions of Theorem 3.3, P_t is strongly Feller.

Outline of the proof:
In order to get strongly Feller property, we should prove

$$\nabla P_t g(x) \leq C(x)\|g\|_\infty.$$

In fact, for any $\xi \in \mathbb{R}^n$,

$$\nabla_\xi P_t g(x) = \nabla_\xi E g(X_t^x) = E\nabla g(X_t^x) J_t \xi,$$

where J_t satisfies

$$\begin{cases} dJ_t = \nabla b(X_t) J_t dt, \\ J_0 = I. \end{cases}$$

Itô formula implies

$$D_V X_t = J_t \int_0^t \int_{\mathbb{R}_0^n} J_s^{-1} V(s,z) N(dz, ds).$$

Choose $V(s,z) = h(z) J_s \xi$ with nonnegative function h. Then

$$D_V X_t = \int_0^t \int_{\mathbb{R}_0^n} h(z) N(dz, ds) J_t \xi.$$

Let $H_t = \int_0^t \int_{\mathbb{R}_0^n} h(z) N(dz, ds)$.

If $H_t > 0, a.s.$, then $H_t^{-1} D_V X_t = J_t \xi$.

Thus,

$$\begin{aligned}
&\nabla_\xi P_t g(x) = \nabla_\xi E g(X_t^x) = E\nabla g(X_t^x) J_t \xi \\
=&E\nabla g(X_t^x) H_t^{-1} D_V X_t = E H_t^{-1} D_V g(X_t^x) \\
=&E D_V(H_t^{-1} g(X_t^x)) - E D_V(H_t^{-1}) g(X_t^x) \\
=&E g(X_t^x)[-H_t^{-1} R_t - D_V(H_t^{-1})] \\
=&E g(X_t^x)[-H_t^{-1} R_t - H_t^{-2} \int_0^t \int_{\mathbb{R}_0^n} \nabla h(z) \cdot V(s,z) N(dz, ds)].
\end{aligned}$$

Basic properties of H_t^{-1}:

L^p-integrability

Proposition 3.5[Norris]

For $p > 1$, if $t > -p \liminf_{\epsilon \longrightarrow 0} \frac{\ln \epsilon}{\nu(h \geq \epsilon)}$, then $EH_t^{-p} < \infty$.

Exponential Integrability

Proposition 3.6

If $h \in L^1_+(U, \nu(dz))$, h and ν satisfy $\liminf_{\epsilon \longrightarrow 0} \nu(h \geq \epsilon)\epsilon > 0$, then there exists $C > 0$ such that

$$E \exp\{H_t^{-1}\} \leq \exp\{C(t + \frac{1}{t})\}, \quad \forall t > 0.$$

Example 3.7

Let $\rho(z) = \frac{1}{|z|^{n+\alpha}}, 0 < \alpha < 2$, S^{n-1} be the unit spherical surface in $\mathbb{R}^n$.
Open sets B, B_1 and B_2 in S^{n-1} satisfy $\overline{B}_1 \subseteq B$ and $\overline{B}_2 \subseteq B_1$. Set
$U = \{z \in \mathbb{R}^n : 0 < |z| < 1, \frac{z}{|z|} \in B\}$,
$U_1 = \{z \in \mathbb{R}^n : 0 < |z| < \frac{2}{3}, \frac{z}{|z|} \in B_1\}$,
$U_2 = \{z \in \mathbb{R}^n : 0 < |z| < \frac{1}{3}, \frac{z}{|z|} \in B_2\}$.
Take functions:
$\phi \in C_b^1(S^{n-1}), I_{B_2} \leq \phi \leq I_{B_1}$;
$\psi \in C_b^1(0,1), I_{(0,\frac{1}{3})} \leq \psi \leq I_{(0,\frac{2}{3})}$;
$h(z) = |z|^\beta \psi(|z|)\phi(\frac{z}{|z|}), \beta > \frac{\alpha}{2} + 1$,
then $h \in \mathcal{H}_\rho$ and conditions of Theorem 3.5 hold.

Set

$$\mathcal{H}'_\rho = \Big\{ h \in L^1_+(U, \nu(dz)) \cap L^2(U, \nu(dz)) \Big| h \in C^1_b(U),$$
$$\frac{h}{dist(\cdot, \partial U)} \in L^2(U, \nu(dz))$$
$$\text{and } \frac{|\nabla(h\rho)|}{\rho} \le Ch(z),\ \ C > 0 \Big\}.$$

Young's Inequality

Theorem 3.8

If $b(x) \in C_b^2(\mathbb{R}^n)$, $h \in \mathcal{H}'_\rho$ and $\liminf_{\epsilon \longrightarrow 0} \nu(h \geq \epsilon)\epsilon > 0$, then for any $\delta > 0$ and bounded nonnegative test function g,

$$\begin{aligned} \nabla_\xi P_t g(x) &\leq \delta \Big\{ P_t(g \log g)(x) - P_t g(x) \log(P_t g(x)) \Big\} \\ &\quad + \delta \exp\left\{ \frac{C(t, b, \xi, \nu, h, \rho)}{\delta} \right\} P_t g(x), \end{aligned}$$

where $C(t, b, \xi, \nu, h, \rho)$ is a constant depending on t, b, ξ, ν, h and ρ.

Harnack Inequality

Theorem 3.9

If $b(x) \in C_b^2(\mathbb{R}^n)$, $h \in \mathcal{H}'_\rho$ and $\liminf_{\epsilon \to 0} \nu(h \geq \epsilon)\epsilon > 0$, then for any $\alpha > 1$ and bounded nonnegative test function g,

$$(P_t g)^\alpha(x) \leq \exp\{\alpha C(t, b, y, \nu, h, \rho)\} P_t g^\alpha(x + y), \quad x, y \in \mathbb{R}^n,$$

where $C(t, b, y, \nu, h, \rho)$ is a constant depending on t, b, y, ν, h and ρ.

Outline

Nonlinear SDEs Driven by Multiplicative Jump Processes

Consider the following equation

$$\begin{cases} dX_t = b(X_{t-})dt + \int_{\mathbb{R}_0^n} f(X_{t-}, z)\widetilde{N}(dz, dt), \\ X_0 = x, \end{cases} \tag{6}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}_0^n \to \mathbb{R}^n$ are measurable.

Hypothesis:
(H4.1) $b \in C_b^2(\mathbb{R}^n)$;
(H4.2) Partial derivatives $f_1'(\cdot, z)$, $f_2'(x, \cdot)$ and $f_{11}''(\cdot, z)$ are bounded on$\mathbb{R}^n \times U$, and satisfy

$$\sup_x |f_1'(x,\cdot)| \text{and} \sup_x |f_{11}''(x,\cdot)| \in L^2(U, \nu(dz));$$

(H4.3) Partial derivatives f_{21}'' and f_{22}'' are bounded on $\mathbb{R}^n \times U$;
(H4.4) $|V(s,z)| \le k(s)h(z)$, where $k : [0, T] \to \mathbb{R}$ are bounded, $h : U \to \mathbb{R}^+$ are continuous bounded function and $h \in \bigcap_{p=1}^{4} L^p(U, \nu(dz))$

Existence of L^2-Derivative:

Theorem 4.1

If (H4.1)-(H4.4) hold, then the solution of equation (6) has an L^2-derivative and the derivative solves the following equation

$$\begin{cases} dD_V X_t = \nabla b(X_t-) D_V X_{t-} dt + \int_{\mathbb{R}_0^n} f_1'(X_{t-}, z) D_V X_{t-} \widetilde{N}(dz, dt) \\ \qquad\qquad + \int_{\mathbb{R}_0^n} f_2'(X_{t-}, z) V(t, z) N(dz, dt), \\ D_V X_0 = 0. \end{cases} \tag{7}$$

- Set

$$Mf = (f_2')^{-1}(I + f_1'), \qquad M = \|Mf\|_\infty,$$

$$\mathcal{G}_\rho = \left\{ h \in \bigcap_{p=1}^{4} L^p(U, \nu(dz)) \Big| h \geq 0, \frac{|\nabla(h\rho)|}{\rho} \in L^2(U, \nu(dz)) \right.$$

$$\left. \text{and } \frac{h}{dist(\cdot, \partial U)} \in L^2(U, \nu(dz)) \right\}.$$

- Non-degeneracy Condition
(H4.5) $\sup_{x \in \mathbb{R}^n, z \in U} \|(I + f_1'(x,z))^{-1}\| < \infty.$

- Jaccobi Matrix

$$\begin{cases} dJ_t = \nabla b(X_t-)J_{t-}dt + \int_{\mathbb{R}_0^n} f_1'(X_{t-}, z)J_{t-}\widetilde{N}(dz, dt), \\ J_0 = I. \end{cases} \tag{8}$$

Inverse Matrix

$$\begin{cases} dJ_t^{-1} = -J_{t-}^{-1}\nabla b(X_{t-})dt - J_{t-}^{-1}\int_{\mathbb{R}_0^n}(I + f_1'(X_{t-}, z))^{-1}f_1'(X_{t-}, z)\widetilde{N}(dz, \\ \qquad\qquad +J_{t-}^{-1}\int_{\mathbb{R}_0^n}(I + f_1'(X_{t-}, z))^{-1}(f_1'(X_{t-,z})^2\nu(dz)dt, \\ J_0^{-1} = I. \end{cases} \tag{9}$$

- When (H4.4)-(H4.4) and (H4.5) hold, both of equation (8) and (9) admit respective solutions.

Theorem 4.2

Assume that (H4.1)-(H4.3) and (H4.5) hold. If $h \in \mathcal{G}_\rho$, $\theta > -\infty$, Mf and $(Mf)'_2$ are bounded, then there exists a constant $C = C(t, h, \nu)$, for any $\xi \in \mathbb{R}^n$, $t > -4\theta$ and bounded test function g

$$|\nabla_\xi P_t g(x)| \leq C\|g\|_\infty \|\xi\| \Bigg\{ \exp\left\{ t\|\nabla b\|_\infty + t\int_{\mathbb{R}_0^n} \sup_x |f_1'(x,z)|^2 \nu(dz) \right\} \times \sqrt{nM^2 \|\frac{\nabla(h\rho)}{\rho}\|_{L^2}^2 + n\|(Mf)_2'\|_\infty^2 \|h\|_{L^2}^2 + M\|\nabla h\|_\infty} \times \exp\left\{ 2t\|\nabla b\|_\infty + t\int_{\mathbb{R}_0^n} \sup_x |f_1'(x,z)|^2 \nu(dz) \right\} \sqrt{t\|h\|_{L^2} + t^2\|h\|_{L^1}^2} \Bigg\}.$$

Corollary 4.3

Under the conditions of Theorem 4.2, P_t is strongly Feller.

Outline of proof

The key is to find a proper direction V. By equation (7), (8) and (9), Itô formula implies

$$D_V X_t = J_t \int_0^t \int_{\mathbb{R}_0^n} J_{s-}^{-1}(I + f_1'(X_{s-}, z))^{-1} f_2'(X_{s-}, z) V(s, z) N(dz, ds).$$

Choose

$$V(s, z) = (f_2'(X_{s-}, z))^{-1}(I + f_1'(X_{s-}, z)) J_{s-} h(z) \xi.$$

Then $D_V X_t = H_t J_t \xi$, where $H_t = \int_0^t \int_U h(z) N(dz, ds)$ with nonnegative function h vanishing off U.

The reminder of the proof is similar to Theorem3.3.

Example 4.4

Let $\rho(z) = \frac{1}{|z|^{n+\alpha}}$ with $0 < \alpha < 2$. Denote the unit spherical surface in $\mathbb{R}^n$ by S^{n-1}. Open sets B, B_1 and B_2 in S^{n-1} satisfy $\overline{B}_1 \subseteq B$ and $\overline{B}_2 \subseteq B_1$.
Set $U = \{z \in \mathbb{R}^n : 0 < |z| < \delta, \frac{z}{|z|} \in B\}$,
$U_1 = \{z \in \mathbb{R}^n : 0 < |z| < \frac{2}{3}\delta, \frac{z}{|z|} \in B_1\}$,
$U_2 = \{z \in \mathbb{R}^n : 0 < |z| < \frac{1}{3}\delta, \frac{z}{|z|} \in B_2\}$.
Take functions:
$\phi \in C_b^1(S^{n-1}), I_{B_2} \leq \phi \leq I_{B_1}$,
$\psi \in C_b^1(0,1), I_{(0,\frac{1}{3}\delta)} \leq \psi \leq I_{(0,\frac{2}{3}\delta)}$,
$h(z) = |z|^\beta \psi(|z|)\phi(\frac{z}{|z|}), \beta > \frac{\alpha}{2} + 1$,
$f(x,z) = g(x)z$, $g \in C_c^2(\mathbb{R}^n), g > 0$.
Then for δ sufficiently small, the conditions of Theorem 4.2 hold.

Set

$$\mathcal{G}'_\rho = \left\{ h \in \bigcap_{p=1}^{4} L^p(U, \nu(dz)) \Big| h \geq 0, \nabla h \in C_b^1(U), \frac{|\nabla(h\rho)|}{\rho} \leq Ch(z) \right.$$
$$\left. \text{and } \frac{h}{dist(\cdot, \partial U)} \in L^2(U, \nu(dz)) \right\}.$$

Young's Inequality:

Theorem 4.5

Assume (H4.1)-(H4.3) and (H4.5) hold. If $h \in \mathcal{G}'_\rho$, $\liminf_{\epsilon \to 0} \nu(h \geq \epsilon)\epsilon > 0$, Mf and $(Mf)'_2$ are bounded, then for any $\delta > 0$ and bounded nonnegative test function g,

$$\begin{aligned}\nabla_\xi P_t g(x) \leq & \delta\Big\{P_t(g\log g)(x) - P_t g(x)\log(P_t g(x))\Big\} \\ & + \delta \exp\left\{\frac{C(t,b,\xi,\nu,h,\rho,f)}{\delta}\right\} P_t g(x),\end{aligned}$$

where $C(t,b,\xi,\nu,h,\rho,f)$ is a constant depending on t, b, ξ, ν, h, ρ and f.

Harnack Inequality:

Theorem 4.6

Under the conditions of Theorem 4.5, for any $\alpha > 1$ and bounded nonnegative test function g,

$$(P_t g)^\alpha(x) \le \exp\{\alpha C(t, b, y, \nu, h, \rho)\} P_t g^\alpha(x + y), \quad x, y \in \mathbb{R}^n,$$

where the constant $C(t, b, y, \nu, h, \rho, f)$ only depends on t, b, y, ν, h, ρ and f.

Acknowledgments:

Thank you !