Asymptotic Behavior for Extinction Probability of the Interacting Branching Collision Process

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- Introduction (IBCP)

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- Known Results: Revisited

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- Main Results: New

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- Main Results: New
- Ideas of Proofs

1. Introduction: IBCP

Def. 1 A conservative q-matrix $Q = \{q_{ij},\ i, j \in Z_+\}$ is called an Interacting Branching Collision q-matrix (IBC q-matrix) if it takes the form:

$$q_{ij} = \begin{cases} \frac{i(i-1)}{2} a_{j-i+2} + i b_{j-i+1} & \text{if } j \geq i-2,\ i \geq 2, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $a_j \geq 0 \quad (j \neq 2)$ and $-a_2 = \sum_{j \neq 2} a_j < +\infty$, together with $a_0 > 0$ and $\sum_{j=3}^{\infty} a_j > 0$. Also

$$b_j \geq 0 \quad (j \neq 1) \quad \text{and} \quad -b_1 = \sum_{j \neq 1} b_j < +\infty, \tag{2}$$

together with $b_0 > 0$, $b_{-1} = 0$ and $\sum_{j=2}^{\infty} b_j > 0$.

1. Introduction: IBCP

Def. 2 An Interacting Branching Collision Process (IBCP) is a Z_+-valued CTMC whose transition function $P(t)$ satisfies the forward equation

$$P'(t) = P(t)Q \tag{3}$$

where Q is an IBC q-matrix.

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Def. 2 An Interacting Branching Collision Process (IBCP) is a Z_+-valued CTMC whose transition function $P(t)$ satisfies the forward equation

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where Q is an IBC q-matrix.

We see that

$$Q = Q^b + Q^c$$

where Q^b and Q^c are the conservative MBP and MCP q-matrices, respectively and thus IBCP has 2 components: MBP and MCP. The former one is well-known while the latter can be seen Chen et al JAP (2004).

1. Introduction: IBCP:

The first component is an MBP whose properties can be analysed by using the generating function of the sequence $\{b_j,\ j \geq 0\}$:

$$B(s) = \sum_{j=0}^{\infty} b_j s^j\,, \qquad |s| \leq 1.$$

Note that $B(0) = b_0 > 0$ and $B(1) = 0$.

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Note that $B(0) = b_0 > 0$ and $B(1) = 0$.

Also $B'(1) = \sum_{j=1}^{\infty} j b_{j+1} - b_0$ satisfies $-\infty < B'(1) \leq +\infty$.

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Lemma 1.1 The equation $B(s) = 0$ has at most two distinct roots in $[0,1]$. More specifically, if $B'(1) \leq 0$ then $B(s) > 0$ for all $s \in [0,1)$ and 1 is the only root of the equation $B(s) = 0$ in $[0,1]$, while if $B'(1) > 0$ (including $B'(1) = +\infty$) then $B(s) = 0$ has an additional root q_b satisfying $0 < q_b < 1$ such that $B(s) > 0$ for $0 \leq s < q_b$ and $B(s) < 0$ for $q_b < s < 1$.

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Moreover, $B(s) = 0$ does not have any other roots in the unit complex disk.

1. Introduction: IBCP

The second component is an MCP whose properties can be analysed by using the generating function of the sequence $\{a_j,\ j \geq 0\}$:

$$A(s) = \sum_{j=0}^{\infty} a_j s^j \,, \qquad |s| \leq 1.$$

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This satisfies $A(0) = a_0 > 0$ and $A(1) = 0$.

Also

$$A'(1) = \sum_{j=1}^{\infty} j a_{j+2} - 2a_0 - a_1$$

satisfies

$$-\infty < A'(1) \leq +\infty.$$

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Lemma 2.1 The equation $A(s) = 0$ has at most two distinct roots in $[0, 1]$. More specifically, if $A'(1) \le 0$ then $A(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $A(s) = 0$ in $[0, 1]$, while if $A'(1) > 0$ (including $A'(1) = +\infty$) then $A(s) = 0$ has an additional root q_c satisfying $0 < q_c < 1$ such that $A(s) > 0$ for $0 \le s < q_c$ and $A(s) < 0$ for $q_c < s < 1$.

The equation $A(s) = 0$ has a unique root η_c in $(-1, 0)$. Moreover, $A(s) = 0$ does not have any other roots in the unit complex disk.

2. Known Results: Revisited

Regularity, Uniqueness and PDE

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Let $\{Z(t),\ t \geq 0\}$ be the unique IBCP and let

$$P(t) = \{p_{ij}(t)\}$$

and

$$R(\lambda) = \{r_{ij}(\lambda)\}$$

denote its transition function and resolvent, respectively.

2. Known Results: Revisited

Theorem 2.1 (PDF) Suppose $P(t)$, $R(\lambda)$ are the Q-function and Q-resolvent of IBCP, respectively. Then

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{A(s)}{2}\frac{\partial^2 F_i(t,s)}{\partial s^2} + B(s)\frac{\partial F_i(t,s)}{\partial s}$$

and

$$\lambda G_i(\lambda,s) - s^i = \frac{A(s)}{2}\frac{\partial^2 G_i(\lambda,s)}{\partial s^2} + B(s)\frac{\partial G_i(\lambda,s)}{\partial s}$$

2. Known Results: Revisited

where

$$F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j, \quad (i \geq 2),$$

and

$$G_i(\lambda,s) = \sum_{j=0}^{\infty} r_{ij}(\lambda)s^j, \quad (i \geq 2).$$

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Theorem 2.3 (Uniqueness) There always exists only one Q-function which satisfies the forward equations. That is that there always exists only one IBCP.

2. Known Results: Revisited

Let $\{Z(t),\ t \geq 0\}$ be the unique IBCP and define the extinction time τ by

$$\tau = \begin{cases} \inf\{t > 0,\ Z(t) = 0\} & \text{if } Z(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } Z(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$a_i = P\{\tau < +\infty | Z(0) = i\}$$

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Theorem 2.4 If $A'(1) \leq 0$ and $B'(1) \leq 0$, then

$$a_i \equiv 1 \ \ (i \geq 1)$$

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Theorem 2.5 If $A'(1) < 0$ and $0 < B'(1) < +\infty$ then

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Remaining case: $A'(1) = 0$ and $0 < B'(1) < +\infty$

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we need to introduce a "testing" function

$$H(y) = \exp\left\{2\int_0^y \frac{B(x)}{A(x)}dx\right\}$$

which possesses many interesting and important properties (but omitted here).

2. Known Results: Revisited

Now define

$$J = \int_{\eta_c}^{1} \frac{H(y)}{A(y)} dy$$

and

$$J_0 = \int_{0}^{1} \frac{H(y)}{A(y)} dy$$

then either $0 < J < +\infty$ or $J = +\infty$.
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Note that Checking J_0 is easier.

2. Known Results: Revisited

Theorem 2.6 Suppose $A'(1) = 0$ and $0 < B'(1) < \infty$.

(i) If $J_0 = +\infty$, then

$$a_i = 1 \quad (i \geq 1)$$

.

(ii) If $J_0 < \infty$ then

$$a_i = J^{-1} \cdot \int_{\eta_c}^{1} \frac{y^i H(y)}{A(y)} dy, \quad i \geq 1$$

.

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The following conclusion is useful since it reduces the possibly hard job in checking of J, or even J_0.

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Theorem 2.7 Suppose $A'(1) = 0, 0 < B'(1) < +\infty$ and $A''(1) < \infty$.

(i) If $A''(1) \geq 4B'(1)$ then $J_0 = +\infty$ and thus

$$a_i = 1$$

.

(ii) If $A''(1) < 4B'(1)$ (including $B'(1) = +\infty$) then $J_0 < \infty$ and thus $a_i < 1$ and

$$a_i = J^{-1} \cdot \int_{\eta_c}^{1} \frac{y^i H(y)}{A(y)} dy, \quad i \geq 1$$

.

2. Known Results: Revisited

Recall IBCP is irregular iff

$$A'(1) > 0$$

or, equivalently, iff

$$q_c < 1$$

2. Known Results: Revisited

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An irregular IBC q-matrix Q is called super-explosive if

$$q_b < q_c < 1$$

critical-explosive if

$$q_b = q_c < 1$$

or sub-explosive if

$$q_c < q_b \leq 1$$

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Theorem 2.8 If $q_b = q_c$, then $a_i = q_b^i$.

The super-explosive case is also not difficult.

Theorem 2.9 If $q_b < q_c < 1$ (super-explosive). Then the extinction probability a_i starting from $i \geq 1$, is

$$a_i = \frac{\int_{\eta_c}^{q_c} \frac{y^i H(y)}{A(y)} dy}{\int_{\eta_c}^{q_c} \frac{H(y)}{A(y)} dy}. \tag{8}$$

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2. Known Results: Revisited

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Theorem 2.10 Suppose that $q_c < q_b \leq 1$ (sub-explosive). Further assume

$$A'(q_c) + 2B(q_c) = 0$$

Then

$$a_i = q_c^i + i\sigma q_c^{n-1} \tag{10}$$

where the positive constant σ is independent of i and given by

$$\sigma = -\frac{B(q_c)}{B'(q_c)}.$$

2. Known Results: Revisited

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Theorem 2.11 Suppose the IBC q-matrix Q is sub-explosive and

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Then

$$a_i = \frac{\int_{\eta_c}^{q_c} \frac{y^i B'(y) - i y^{i-1} B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy}{\int_{\xi_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy}.$$

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For Definition of $A_1(x)$ and $B_1(x)$, see below

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By **Lemmas 1.1 and 3.1** we know $q_c < q_b \leq 1$ (sub-explosive)implies

$$A'(q_c) < 0$$

and

$$B(q_c) > 0$$

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$$A'(q_c) < 0$$

and

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One thus could find the smallest positive integer k such that

$$kA'(q_c) + 2B(q_c) \leq 0.$$

2. Known Results: Revisited

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We may get (details omitted including the definitions of $D_{m,k}$ etc.

2. Known Results: Revisited

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Theorem 2.12 Suppose that Q is a sub-explosive IBC-q-matrix satisfying

$$A'(q_c) + 2B(q_c) > 0$$

and that

$$-2B(q_c)/A'(q_c)$$

is not an integer. Let

$$m = \min\{k \geq 1; kA'(q_c) + 2B(q_c) < 0\}$$

.

2. Known Results: Revisited

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$$a_i = \frac{\sum_{k=0}^{m\wedge i} \frac{i!}{(i-k)!} \int_{\eta_c}^{q_c} \frac{y^{i-k} D_{m,k}(y)}{A_m(y)} e^{H_m(y)} dy}{\int_{\eta_c}^{q_c} \frac{D_{m,0}(y)}{A_m(y)} e^{H_m(y)} dy}. \quad (12)$$

2. Known Results: Revisited

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Remark: If

$$-2B(q_c)/A'(q_c)$$

is an integer, the problem is much simpler.

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Q is regular and hence $\rho_c = 1$. Then only need to consider the case that $C'(1) = 0$ and $C''(1) < 4B'(1) < +\infty$

Q is irregular and hence $\rho_c < 1$. Then only need to consider the case that $\rho_c \neq \rho_b$

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Theorem 3.1 If $C'(1) = 0$ and $C''(1) < 4B''(1) < +\infty$, then the extinction probability a_n satisfies

$$a_n \sim kn^{1-\alpha} \quad \text{as } n \to +\infty$$

where k is a constant (independent of n) and $\alpha = \frac{4B'(1)}{C''(1)} > 1$.

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where k is a constant (independent of n) and $\alpha = \frac{4B'(1)}{C''(1)} > 1$.

Next, consider the case Q is irregular and hence $\rho_c < 1$. There are a few sub-cases as follows.

3. Main Results: New

Theorem 3.2 If $\rho_b < \rho_c < 1$, then the extinction probability of the IBCP, starting from $n \geq 1$, denoted by $\{a_n\}$, possesses the following asymptotic behavior

$$a_n \sim kn^{-\alpha}\rho_c^n \quad (\text{ as } n \to +\infty)$$

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where k is a constant and $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$.

Repeat: For this case $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$

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or saying another way, the extinction probability of the

IBCP, $\{a_n\}$, possesses the asymptotic behavior

$$a_n \sim k n^{-\alpha} \rho_c^n \quad (\text{ as } n \to +\infty)$$

where k is a constant and $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} = -1$.

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Note: For this case $\alpha = -1$.

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where k is a constant and $-1 < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$.

Note: For this case $-1 < \alpha < 0$.

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where k is a constant and $-(m+1) < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < -m$.

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Question: $\alpha = 0$?

Answer: $\rho_c = \rho_b$. Recall Th 2.8 $(a_n = \rho_c^n \equiv n^0\rho_c^n)$

4. Ideas of Proofs

Main steps in Proving Th.3.2 **Step 1: Show that if $\rho_c < 1$, then $\exists$ some constant k such that**

$$A(y) = \exp\left\{\int_0^y \frac{2B(x)}{C(x)}\mathrm{d}x\right\} \sim k\left(\rho_c - y\right)^{\alpha} \qquad \textbf{as } y \to \rho_c^-,$$

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Indeed, let

$$g(x) = \frac{2B(x)(\rho_c - x)}{C(x)}$$

then $g(x)$ is "qualified" to be expanded in the interval $[0, \rho_c]$ as a power series $g(x) = \sum_{k=0}^{\infty} g_k x^k$

4. Ideas of Proofs

Now for $0 < y < \rho_c$ we have

$$\int_0^y \frac{2B(x)}{C(x)}\mathrm{d}x = \int_0^y \frac{g(x)}{\rho_c - x}\mathrm{d}x = \sum_{k=0}^{\infty} g_k \int_0^y \frac{x^k}{\rho_c - x}\mathrm{d}x$$

$$= \quad g_0 \int_0^y \frac{\mathrm{d}x}{\rho_c - x} + \sum_{k=1}^{\infty} g_k \int_0^y \frac{\rho_c^k + \sum_{m=1}^{k}(-1)^m \binom{k}{m}(\rho_c - x)^m \rho_c^{k-}}{\rho_c - x}$$

$$= \quad \left(\sum_{k=0}^{\infty} g_k \rho_c^k\right) \int_0^y \frac{\mathrm{d}x}{\rho_c - x} + \sum_{k=1}^{\infty} g_k \sum_{m=1}^{k} (-1)^m \binom{k}{m} \rho_c^{k-m} \int_0^y (\rho_c$$

$$= \quad J_1 + J_2$$

where the meaning of J_1 and J_2 are self-explained.

4. Ideas of Proofs

Easy to see

$$J_1 = \left(\sum_{k=1}^{\infty} g_k \rho_c^k\right) \int_0^y \frac{\mathrm{d}x}{\rho_c - x} = g(\rho_c) \int_0^y \frac{\mathrm{d}x}{\rho_c - x}$$

where $g(\rho_c) = \lim_{x \to \rho_c^+} \frac{2B(x)(\rho_c - x)}{C(x)} = -\frac{2B(\rho_c)}{C'(\rho_c)}$ which is finite.

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By some algebra and applying the integral mean-valued theorem we are able to show that J_2 can be written as as a constant which is independent of y. This finishes the proof of Step 1.

4. Ideas of Proofs

Main steps in Proving Th.3.2 **Step 2: Show that there exists a constant k such that**

$$I_1^{(n)} = \int_0^{\rho_c} \frac{y^n A(y)}{C(y)} \mathrm{d}y = k \cdot \int_0^{\rho_c} y^n \left(\rho_c - y\right)^{\alpha-1} \mathrm{d}y$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$ since both $B(\rho_c)$ and $C'(\rho_c)$ are negative under the conditions of this Theorem.

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where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$ **since both** $B(\rho_c)$ **and** $C'(\rho_c)$ **are negative under the conditions of this Theorem.**

This step could be easily proved by applying the results obtained in the first step together with some algebras.

4. Ideas of Proofs

Main steps in Proving Th.3.2 **Step 3: Show that**

$$I_1^{(n)} \sim kn^{-\alpha}\rho_c^n.$$

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Just note that

$$\int_0^{\rho_c} y^n \left(\rho_c - y\right)^{\alpha-1} \mathrm{d}y = \rho_c^{n+\alpha} \cdot \int_0^1 x^n (1-x)^{\alpha-1} \mathrm{d}x$$

which is just

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We could finishes the proof of Step 3 by noting that our $\alpha > 0$.

4. Ideas of Proofs

Main steps in Proving Th.3.2 **Final Step 4: Show that**

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Just note that $a_n \sim I_1^{(n)} + I_2^{(n)}$ where $I_2^{(n)} = \int_{\xi_c}^{0} \frac{y^n A(y)}{C(y)} \mathrm{d}y$.

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$I_2^{(n)}$ could be similarly analyzed as $I_1^{(n)}$ as above. Then use the fact that $|\xi_c| < \rho_c$ to prove the conclusion. The details for this last step omitted.

4. Ideas of Proofs

Main ideas in proving other theorems. Similarly, but details different and omitted.

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A key point in proving the last theorem, in particular, is that (by recalling the definitions of $A_n(s)$ and $B_n(s)$) we have

$$\frac{B_m(s)}{A_m(s)} = \frac{B_{m-1}(s)}{A_{m-1}(s)} + \frac{A'_{m-1}(s)}{A_{m-1}(s)} - \frac{B'_{m-1}(s)}{B_{m-1}(s)}.$$

By integrating the above from 0 to y and then raising to the exponential, we could decrease m to $m-1$. By repeating this procedure, we could reduce the problem into a question similar to Theorem 3.2.