Functional inequalities for subelliptic diffusion operators via curvature bounds

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$$\Gamma(f,g) = \frac{1}{2}\left(\Delta(fg) - f\Delta g - g\Delta f\right) = \langle \nabla f, \nabla g\rangle$$

and

$$\Gamma_2(f,g) = \frac{1}{2}\left(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\right).$$

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Theorem

We have $\mathbf{Ric} \geq \rho$ *and* $\mathbf{dim}\mathbb{M} \leq n$ *if and only if for every smooth* f*,*

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This leads to the notion of intrinsic curvature-dimension bounds for diffusion operators. To be satisfied, the curvature-dimension inequality requires some form of ellipticity.

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- A n-dimensional complex sub-bundle $\mathcal{C}$ of the complexified bundle $\mathbb{C}T\mathbb{M}$ that satisfies $[\mathcal{C}, \mathcal{C}] \subset \mathcal{C}$ and $\mathcal{C} \cap \bar{\mathcal{C}} = 0$;

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The CR manifold is said to be strictly pseudo-convex if the bilinear form

$$g_\theta(X, Y) = d\theta(X, JY)$$

is positive definite on $\mathbf{Ker}(\theta)$.

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The sub-Laplacian Δ on $\mathbb{M}$ is the diffusion operator on $\mathbb{M}$ which is symmetric with respect to the volume form $\theta \wedge (d\theta)^n$ and such that

$$\Gamma(f) = \|\nabla^{\mathcal{H}} f\|^2.$$

where $\nabla^{\mathcal{H}} f$ is the horizontal gradient of f.

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Is there a notion of curvature dimension bounds for Δ ?

The Heisenberg group

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We have a CR Lie group with horizontal space $\mathbf{span}(X, Y)$ and Reeb vector field Z.

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$$\begin{aligned}\Gamma_2(f) =& (X^2 f)^2 + (Y^2 f)^2 + \frac{1}{2}((XY + YX)f)^2 + \frac{1}{2}(Zf)^2 \\ &- 2(Xf)(YZf) + 2(Yf)(XZf).\end{aligned}$$

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The mixt term $-2(Xf)(YZf) + 2(Yf)(XZf)$ prevents to find any lower bound on this quantity involving $\Gamma(f)$ only !

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Theorem

For every $\nu > 0$,

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{2}(Lf)^2 - \frac{1}{\nu}\Gamma(f) + \frac{1}{2}\Gamma^Z(f).$$

The Tanaka-Webster connection on a CR manifold

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Theorem

Let $\mathbb{M}$ be a CR Sasakian manifold. We have $\mathbf{dim}(H) \leq d$ *and* $\mathbf{Ric}_\nabla \geq \rho_1$ *if and only if for every* $\nu > 0$,

$$\Gamma_2(f) + \nu \Gamma_2^T(f) \geq \frac{1}{d}(\Delta f)^2 + \left(\rho_1 - \frac{1}{\nu}\right)\Gamma(f) + \frac{d}{4}(Tf)^2,$$

where

$$2\Gamma_2^T(f) = \Delta(Tf)^2 - 2TfT\Delta f$$

Generalized curvature dimension inequality

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In the context of CR manifolds this commutation is equivalent to the fact that the torsion is vertical (Sasakian assumption).

Generalized curvature dimension inequality

Definition

We say that L satisfies the generalized-curvature inequality $CD(\rho_1, \rho_2, \kappa, d)$ if for every $\nu > 0$,

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2 \Gamma^Z(f).$$

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$CD(\rho_1, \rho_2, \kappa, d)$ is the linearization of

$$\Gamma_2(f) + 2\sqrt{\kappa \Gamma(f) \Gamma_2^Z(f)} \geq \frac{1}{d}(Lf)^2 + \rho_1 \Gamma(f) + \rho_2 \Gamma^Z(f).$$

Examples

We have the following general class of examples:

- Let $\mathbb{M}$ be a n-dimensional complete Riemannian manifold wiose Ricci curvature is bounded from below by ρ. The Laplacian of $\mathbb{M}$ satisfies the curvature dimension inequality $\mathrm{CD}(\rho, 0, 0, d)$ with $\Gamma^Z = 0$.

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- Two-step nilpotent Lie groups.
- Orthonormal bundles over Riemannian manifolds.
- Fibrations
- Infinite dimensional examples (Baudoin-Gordina-Melcher, Trans. AMS 2012)

Li-Yau inequality

Theorem (Baudoin-Garofalo, 2011)

Assume that L satisfies the generalized curvature inequality $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \geq 0$, $\rho_2 > 0$, $\kappa \geq 0$ and $0 < d < \infty$, then

$$\Gamma(\ln p_t) + \frac{2\rho_2}{3} t \Gamma^Z(\ln p_t) \leq (1 + \frac{3\kappa}{2\rho_2}) \frac{Lp_t}{p_t} + \frac{d\left(1 + \frac{3\kappa}{2\rho_2}\right)^2}{2t}.$$

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$$p(x, y, s) \leq p(x, z, t) \left(\frac{t}{s} \right)^{\frac{D}{2}} \exp \left(\frac{D}{d} \frac{d(y, z)^2}{4(t - s)} \right),$$

where

$$D = d \left(1 + \frac{3\kappa}{2\rho_2} \right).$$

Gaussian bounds

Theorem (Baudoin-Bonnefont-Garofalo, 2011)

Assume that L satisfies the generalized curvature inequality $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \geq 0$, $\rho_2 > 0$, $\kappa \geq 0$ and $0 < d < \infty$. For any $0 < \epsilon < 1$, there exists a constant $C = C(d, \kappa, \rho_2, \epsilon) > 0$, which tends to ∞ as $\epsilon \to 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has

$$\frac{\exp\left(-\frac{d(x,y)^2}{(4-\epsilon)t}\right)}{C\mu(B(x,\sqrt{t}))} \leq p(x,y,t) \leq C\frac{\exp\left(-\frac{d(x,y)^2}{(4+\epsilon)t}\right)}{\mu(B(x,\sqrt{t}))}.$$

Subelliptic Myers theorem

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Theorem (Baudoin-Garofalo, 2011)

If the inequality $CD(\rho_1, \rho_2, \kappa, d)$ holds for some constants $\rho_1 > 0, \rho_2 > 0, \kappa > 0$, then the metric space $(\mathbb{M}, d)$ is compact in the metric topology and we have

$$diam\, \mathbb{M} \le 2\sqrt{3}\pi \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2}\left(1 + \frac{3\kappa}{2\rho_2}\right) d};$$

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Riesz transform

Theorem (Baudoin-Garofalo, IMRN 2012)

Assume that L *satisfies the generalized curvature inequality* $CD(\rho_1, \rho_2, \kappa, d)$ *with* $\rho_1 \geq 0$, $\rho_2 > 0$, $\kappa \geq 0$ *and* $0 < d < \infty$. *Let* $1 < p < \infty$. *There exist constants* $A_p, B_p > 0$ *such that*

$$A_p \|(-L)^{1/2} f\|_p \leq \|\sqrt{\Gamma(f)}\|_p \leq B_p \|(-L)^{1/2} f\|_p, \qquad f \in C_0^\infty(\mathbb{M}),$$

Spectral gap inequality

Theorem (Baudoin-Bonnefont, JFA 2012)

Assume that L satisfies the generalized curvature inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ with $\rho_1 > 0$, $\rho_2 > 0$ and $\kappa \geq 0$. The measure μ is finite and the following Poincaré inequality holds:

$$\int_{\mathbb{M}} f^2 d\mu - \left(\int_{\mathbb{M}} f d\mu \right)^2 \leq \frac{\kappa + \rho_2}{\rho_1 \rho_2} \int_{\mathbb{M}} \Gamma(f) d\mu$$

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Modified log-Sobolev inequality

Theorem (Baudoin-Bonnefont, JFA 2012)

Assume that L satisfies the generalized curvature inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ with $\rho_1 > 0$, $\rho_2 > 0$ and $\kappa \geq 0$. If μ is a probability measure, then

$$\int_{\mathbb{M}} f^2 \ln f^2 d\mu - \int_{\mathbb{M}} f^2 d\mu \ln \int_{\mathbb{M}} f^2 d\mu$$

$$\leq \frac{2(\kappa + \rho_2)}{\rho_1 \rho_2} \left(\int_{\mathbb{M}} \Gamma(f) d\mu + \frac{\kappa + \rho_2}{\rho_1} \int_{\mathbb{M}} \Gamma^Z(f) d\mu \right).$$

Modified log-Sobolev inequality

Theorem (Baudoin-Bonnefont, JFA 2012)

Assume that L satisfies the generalized curvature inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ with $\rho_1 > 0$, $\rho_2 > 0$ and $\kappa \geq 0$. If μ is a probability measure, then

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Observe the annoying additional term $\int_{\mathbb{M}} \Gamma^Z(f) d\mu$.

Log-Sobolev inequality

Log-Sobolev inequality

Theorem (Baudoin-Bonnefont, JFA 2012)

Assume that the measure μ *is a probability measure and that* L *satisfies the generalized curvature dimension inequality* $CD(\rho_1, \rho_2, \kappa, \infty)$ *for some* $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$. *Assume moreover that*

$$\int_{\mathbb{M}} e^{\lambda d^2(x_0,x)} d\mu(x) < +\infty,$$

Log-Sobolev inequality

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$$\int_{\mathbb{M}} e^{\lambda d^2(x_0,x)} d\mu(x) < +\infty,$$

for some $x_0 \in \mathbb{M}$ *and* $\lambda > \frac{\rho_1^-}{2}$, *then there is a constant* $\rho_0 > 0$ *such that for every function* $f \in C_0^\infty(\mathbb{M})$,

$$\int_{\mathbb{M}} f^2 \ln f^2 d\mu - \int_{\mathbb{M}} f^2 d\mu \ln \int_{\mathbb{M}} f^2 d\mu \leq \frac{2}{\rho_0} \int_{\mathbb{M}} \Gamma(f) d\mu.$$

Theorem (Baudoin-Bonnefont, JFA 2012)

Assume that the measure μ *is a probability measure, that* L *satisfies the generalized curvature dimension inequality* $CD(\rho_1, \rho_2, \kappa, \infty)$ *for some* $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$ *and that* μ *satisfies the log-Sobolev inequality:*

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Gaussian isoperimetry

Theorem (Baudoin-Bonnefont, JFA 2012)

Assume that the measure μ is a probability measure, that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$ and that μ satisfies the log-Sobolev inequality:

$$\int_{\mathbb{M}} f^2 \ln f^2 d\mu - \int_{\mathbb{M}} f^2 d\mu \ln \int_{\mathbb{M}} f^2 d\mu \leq \frac{2}{\rho_0} \int_{\mathbb{M}} \Gamma(f) d\mu, \qquad (1)$$

Let A be a set of the manifold $\mathbb{M}$ which has a finite perimeter $P(A)$ and such that $0 \leq \mu(A) \leq \frac{1}{2}$, then

$$P(A) \geq \frac{\ln 2}{4\left(3 + \frac{2\kappa}{\rho_2}\right)} \min\left(\sqrt{\rho}_0, \frac{\rho_0}{\sqrt{\rho_1^-}}\right) \mu(A) \left(\ln \frac{1}{\mu(A)}\right)^{\frac{1}{2}}.$$