Bismut Formulae and Applications for Functional SPDEs

Jianhai Bao

Joint work with Feng-Yu Wang and Chenggui Yuan

Swansea University

July 20th, 2012

Outline

- Motivations
- Main Results
- Applications: Gradient Estimate and Harnack Inequality
- Ideas of Proofs for the Main Results

Motivations

The Bismut-type formulae, initiated in Bismut (1984), are powerful tools to derive regularity estimates for the underlying diffusion semigroups. The formulae have been developed and applied in various setting:

- In Da Prato & Zabczyk (1996) for SPDEs driven by cylindrical Wiener processes and Dong & Xie (2010) for semi-linear SPDEs with Lévy noise, using a martingale approach proposed by Elworthy & Li (1994);
- In Wang (2011) for linear SDEs driven by (purely jump) Lévy processes in terms of lower bound conditions of Lévy measures;
- In Bao, Wang and Yuan (2011) and Guillin & Wang (2011) for degenerate SDEs with additive noise, using a coupling technique;

- In Fuhrman (1996), Priola (2006), Wang & Zhang (2011) and Zhang (2010) for degenerate SDEs, utilizing Malliavin Calculus.

However, there are few analogues for functional SPDEs (even for finite-dimensional functional SDEs) with multiplicative noise. For functional SPDEs

- Martingale method used in Elworthy and Li (1994) does not work due to the lack of backward Kolmogrov equation for the segment process;
- The coupling method developed in, e.g., Arnaudon, Thalmaier and Wang (2006), Bao, Wang and Yuan (2011), Guillin & Wang (2011) and Wang & Xu (2010) seems not easy to apply provided that the noise is multiplicative.

In this talk we shall give explicit Bismut-type formulae for a class of functional SPDEs with additive and multiplicative noise, using Malliavin calculus.

Main Results

We first introduce the following notation:

- $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, real separable Hilbert space;
- $W(t)$, an H-valued cylindrical Wiener process with respect to a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$;
- $(\mathcal{L}(H), \|\cdot\|)$, space of linear bounded operators on H, and $(\mathcal{L}_{HS}(H), \|\cdot\|_{HS})$, space of Hilbert-Schmidt operators from H to H;
- For $\tau > 0$, $\mathscr{C} := C([-\tau, 0] \to H)$, space of all H-valued continuous functions f defined on $[-\tau, 0]$, with a uniform norm;
- For a map $h : [-\tau, \infty) \to H$ and $t \geq 0$, let $h_t \in \mathscr{C}$ be the segment of $h(t)$, i.e., $h_t(\theta) = h(t+\theta), \theta \in [-\tau, 0]$.

Main Results

Consider the following semi-linear functional SPDE

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + F(X_t)\}\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \\ X_0 = \xi \in \mathscr{C}, \end{cases} \tag{1}$$

where

(A1) $(A, \mathcal{D}(A))$ is a linear operator on H generating a contractive C_0-semigroup $(e^{tA})_{t\geq 0}$.

(A2) $F : \mathscr{C} \to H$ is Fréchet differentiable such that $\nabla F \in C_b(\mathscr{C} \times \mathscr{C} \to H)$.

(A3) $\sigma : H \to \mathcal{L}(H)$ is Fréchet differentiable such that $\nabla\sigma \in C_b(H \times H \to \mathcal{L}(H))$.

(A4) For any $T > 0$, $\int_0^T \|e^{sA}\sigma(0)\|_{HS}^2 ds < \infty$.

Main Results

By $(A1)-(A4)$, equation (1) has a unique mild solution, denoted by $(X^{\xi}(t))_{t\geq 0}$, starting from the initial data $\xi\in\mathscr{C}$. Note that

- Due to the time-delay the solution $(X^{\xi}(t))_{t\geq 0}$ is not Markovian;
- The segment process $(X_t^{\xi})_{t\geq 0}$ admits strong Markov property.

Let

$$P_t f(\xi) := \mathbb{E} f(X_t^{\xi}), \quad t\geq 0, \xi\in\mathscr{C}, f\in\mathscr{B}_b(\mathscr{C}),$$

where $\mathscr{B}_b(\mathscr{C})$ is the class of all bounded measurable functions on $\mathscr{C}$.

The following two theorems are the main results of this talk, which provide derivative formulae for P_t with additive and multiplicative noise respectively.

Main Results

Theorem

(Additive Noise) Assume that $(A1)-(A4)$ hold with constant $\sigma \in \mathcal{L}(H)$ such that $e^{tA}(H) \subset \sigma(H)$ for $t > 0$, $F(H) \subset \sigma(H)$ and

$$\|\sigma^{-1}\| \leq K \text{ for some constant } K. \tag{2}$$

Then for any $T > \tau$, $u \in C^1([0, T-\tau])$ such that $u(0) = 1$ and $u(T-\tau) = 0$, $\xi, \eta \in \mathscr{C}$ and $f \in C_b^1(\mathscr{C})$,

$$\nabla_\eta P_T f(\xi) = \mathbb{E}\bigg(f(X_T^\xi) \int_0^{T-\tau} \langle \sigma^{-1}(\nabla_{\Upsilon_t} F(X_t^\xi) - \dot{u}(t) e^{tA} \eta(0)), \mathrm{d}W(t) \rangle_H \bigg)$$

holds, where $\Upsilon(t) := u(t) e^{tA} \eta(0) 1_{[0,T-\tau]}(t), t \geq 0$, and $\Upsilon(t) := \eta(t), t \in [-\tau, 0]$.

Main Results

Theorem

(Multiplicative Noise Case) Assume that $(A1)-(A4)$ hold and that $\sigma(x)$ is invertible for $x \in H$. Let $T > \tau$ and $u \in C^1([0, T-\tau])$ be such that $u(t) > 0$ for $t \in [0, T-\tau)$, $u(T-\tau) = 0$ and $\theta_1 := \inf_{t\in[0,T-\tau]}(2+u'(t)) > 0$. Then for any $\xi, \eta \in \mathscr{C}$

(1) The equation

$$\begin{cases} \mathrm{d}Z(t) = \Big\{AZ(t) + \nabla_{Z_t}F(X_t^\xi) - \frac{Z(t)}{u(t)}1_{[0,T-\tau)}(t) \\ \qquad\qquad -\nabla_{Z_t}F(X_t^\xi)1_{[T-\tau,T]}(t)\Big\}\mathrm{d}t + (\nabla_{Z(t)}\sigma(X^\xi(t)))\mathrm{d}W(t), \\ Z_0 = \eta, \end{cases} \tag{3}$$

Main Results

has a unique solution and, in particular, $Z(t) = 0$ a.s. for $t \geq T - \tau$.

(2) If $\theta_2 := \sup_{x \in H} \|\sigma^{-1}(x)\| < \infty$, then

$$\nabla_\eta P_T f(\xi) = \mathbb{E}\bigg(f(X_T^\xi) \int_0^T \Big\langle \sigma^{-1}(X^\xi(t)) \Big\{ \frac{Z(t)}{u(t)} 1_{[0,T-\tau)}(t) + \nabla_{Z_t} F(X_t^\xi) 1_{[T-\tau,T]}(t) \Big\}, \mathrm{d}W(t) \Big\rangle_H \bigg) \tag{4}$$

holds for $f \in C_b^1(\mathscr{C})$.

Remark: For the additive case, we can choose $u(t) = (T - \tau - t)^+ / (T - \tau)$ and for the multiplicative case we can take $u(t) = (T - \tau - t)^+$.

Theorem 1 (Additive Noise) Assume that (A1)-(A4) hold with constant $\sigma \in \mathcal{L}(H)$. Then there exists a constant $C > 0$ such that

(1) For any $T > \tau, \xi, \eta \in \mathscr{C}$ and $f \in \mathscr{B}_b(\mathscr{C})$,

$$|\nabla_\eta P_T f(\xi)|^2 \leq \frac{C}{(T-\tau)\wedge 1} P_T f^2(\xi).$$

(2) For any $T > \tau, \xi, \eta \in \mathscr{C}$ and positive $f \in \mathscr{B}_b(\mathscr{C})$,

$$|\nabla_\eta P_T f(\xi)| \leq \delta(P_T(f\log f) - (P_T f)\log P_T f)(\xi) + \frac{\|\eta\|_\infty^2}{\delta\{(T-\tau)\wedge 1\}} P_T f(\xi \tag{5}$$

Applications: Gradient Estimate and Harnack Inequality

Applying the Young inequality (Proposition 4.1, Guillin and Wang 2011), we have the following Harnack inequality

Corollary

Assume that (A1)-(A4) hold with constant $\sigma \in \mathcal{L}(H)$. Then there exists a constant $C > 0$ such that

$$|P_T f|^\alpha \leq \exp\left[\frac{\alpha C\|\eta\|_\infty^2}{(\alpha-1)\{(T-\tau)\wedge 1\}}\right] P_T|f|^\alpha(\xi+\eta), f \in \mathscr{B}_b(\mathscr{C}), T > \tau,$$

$\xi, \eta \in \mathscr{C}$ holds for any $\alpha > 1$.

Applications: Gradient Estimate and Harnack Inequality

Theorem 2 (Multiplicative Case) Let (A1)-(A4) hold and assume that. $\|\sigma^{-1}\|_\infty := \sup_{x\in H}\|\sigma^{-1}(x)\| < \infty$. Then for any $p > 1$ there exists a constant $C > 0$ such that

$$|\nabla_\eta P_T f(\xi)| \leq \frac{C\|\eta\|_\infty}{1\wedge\sqrt{T-\tau}}(P_T|f|^p)^{1/p}(\xi), f\in\mathscr{B}_b(\mathscr{C}), T>\tau, \xi,\eta\in\mathscr{C}.$$

In particular, P_t is strong Feller for $t > T-\tau$.

Some Remarks

- From the previous Corollary and (Proposition 4.1, Guillin and Wang 2011), we know that entropy estimation (5) plays a key role in establishing the Harnack inequality;
- The entropy estimation seems to be difficult to obtain for the multiplicative noise case. Hence we cannot adopt the same method as in the additive noise case to derive the Harnack inequality;
- In order to establish the Harnack inequality for the multiplicative noise case, we may follow the coupling method as in Wang (2011), and Wang and Yuan (2011).

Ideas of Proof of the Main Results

Let H_a^1 be the class of all adapted process $h = (h(t))_{t\geq 0}$ on H such that $h(0) = 0$,

$$\dot{h}(t) := \frac{\mathrm{d}}{\mathrm{d}t} h(t)$$

exists $\mathbb{P} \times \mathrm{d}t$-a.e. and

$$\mathbb{E} \int_0^T \|\dot{h}(t)\|_H^2 \mathrm{d}t < \infty, \qquad T > 0.$$

For $\epsilon > 0$ and $h \in H_a^1$, let $X^{\xi,\epsilon h}(t)$ solve (1) with $W(t)$ replaced by $W(t) + \epsilon h(t)$, i.e.,

$$\begin{cases} \mathrm{d}X^{\xi,\epsilon h}(t) = \{AX^{\xi,\epsilon h}(t) + F(X_t^{\xi,\epsilon h}) + \epsilon\sigma(X^{\xi,\epsilon h}(t))\dot{h}(t)\}\mathrm{d}t \\ \qquad\qquad +\sigma(X^{\xi,\epsilon h}(t))\mathrm{d}W(t), \\ X_0^{\xi,\epsilon h} = \xi \in \mathscr{C}. \end{cases} \tag{6}$$

Ideas of Proof of the Main Results

If for $h \in H_a^1$
$$D_h X_t^\xi := \frac{\mathrm{d}}{\mathrm{d}\epsilon} X_t^{\xi,\epsilon h}\Big|_{\epsilon=0}$$
exists in $L^2(\Omega \to H; \mathbb{P})$, we call it the Malliavin derivative of X_t^ξ along direction h. Next let
$$\nabla_\eta X_t^\xi := \frac{\mathrm{d}}{\mathrm{d}\epsilon} X_t^{\xi+\epsilon\eta}\Big|_{\epsilon=0}$$
be the derivative process of X_t^ξ along direction $\eta \in \mathscr{C}$. If
$$D_h X_T^\xi = \nabla_\eta X_T^\xi, \quad \text{a.s.,} \tag{7}$$
then for any $f \in C_b^1(\mathscr{C})$
$$\begin{aligned}\nabla_\eta P_T f(\xi) &= \mathbb{E}\nabla_\eta f(X_T^\xi) = \mathbb{E}\nabla_{\nabla_\eta X_T^\xi} f(X_T^\xi)\\ &= \mathbb{E}\nabla_{D_h X_T^\xi} f(X_T^\xi) = \mathbb{E} D_h f(X_T^\xi).\end{aligned}$$

Ideas of Proof of the Main Results

Combining this with the integration by parts formula for D_h, we obtain

$$\nabla_\eta P_T f(\xi) = \mathbb{E}\Big(f(X_T^\xi) \int_0^T \langle \dot{h}(t), \mathrm{d}W(t) \rangle_H \Big).$$

It is easy to see that the key point of the proof is, for given $T > \tau$, $\xi, \eta \in \mathscr{C}$ and $f \in C_b^1(\mathscr{C})$, how to construct an $h \in H_a^1$ such that (7) holds.

Proof of Theorem 1

- Let $h(0) = 0$ and

$$\dot{h}(t) := \sigma^{-1}\{\nabla_{\Upsilon_t} F(X_t^\xi) - \dot{u}(t)e^{tA}\eta(0)\}), \quad t \geq 0,$$

- $\Upsilon(t)$ solves the equation

$$\begin{cases} \mathrm{d}\Upsilon(t) = \{A\Upsilon(t) + \nabla_{\Upsilon_t} F(X_t^\xi) - \sigma\dot{h}(t)\}\mathrm{d}t, & t \geq 0, \\ \Upsilon_0 = \eta. \end{cases} \tag{8}$$

- $\nabla_\eta X^\xi(t) - D_h X^\xi(t)$ also solves (8).
- By the uniqueness, we have $\nabla_\eta X^\xi(t) - D_h X^\xi(t) = \Upsilon(t), t \geq 0$.
- $\nabla_\eta X_T^\xi = D_h X_T^\xi$ as $\Upsilon_T = 0$ according to the choice of u.

Proof of Theorem 2

- By the Itô formula, we have

$$\lim_{t \uparrow T-\tau} Z(t) = 0.$$

- Let

$$h(t) = \int_0^t \sigma^{-1}(X^\xi(s))\Big\{\frac{Z(s)}{u(s)} 1_{[0,T-\tau)}(s) + \nabla_{Z_s} F(X_s^\xi) 1_{[T-\tau,T]}(s)\Big\} \mathrm{d}s, t$$

- $\Gamma(t) := \nabla_\eta X^\xi(t) - D_h X^\xi(t)$ solves the equation

$$\begin{cases} \mathrm{d}\Gamma(t) = \Big\{A\Gamma(t) + \nabla_{\Gamma_t} F(X_t^\xi) - \frac{Z(t)}{u(t)} 1_{[0,T-\tau)}(t) - \nabla_{Z_t} F(X_t^\xi) 1_{[T-\tau,T]}(\\ \qquad\qquad + \Big\}\mathrm{d}t + \nabla_{\Gamma(t)} \sigma(X^\xi(t))\mathrm{d}W(t), \\ \Gamma_0 = \eta. \end{cases}$$

Proof of Theorem 2

- Then for $t \in [0,T]$
$$\begin{cases} \mathrm{d}(\Gamma(t)-Z(t)) = \Big\{A(\Gamma(t)-Z(t)) + \nabla_{\Gamma_t - Z_t}F(X_t^\xi)\Big\}\mathrm{d}t \\ \qquad\qquad\qquad + \nabla_{\Gamma(t)-Z(t)}\sigma(X^\xi(t))\mathrm{d}W(t), \\ \Gamma_0 - Z_0 = 0. \end{cases}$$
- By the Itô formula, (A1)-(A3) and the Burkhold-davis-Gundy inequality, we obtain
$$\mathbb{E}\sup_{s\in[0,t]}\|\Gamma(s)-Z(s)\|_H^2 \le C\int_0^t \mathbb{E}\sup_{s\in[0,r]}\|\Gamma(s)-Z(s)\|_H^2\mathrm{d}r, \quad t\ge 0$$
for some constant $C>0$.
- $\Gamma(t) = Z(t)$ for all $t\in[0,T]$. In particular, $\Gamma_T = Z_T$. Since $Z_T = 0$, one has $\nabla_\eta X_T^\xi - D_h X_T^\xi$.

References

- Bao, J., Wang, F.-Y. and Yuan, C., Derivative Formula and Harnack Inequality for Degenerate Functional SDEs, arXiv:1109.3907v1.
- Bao, J., Wang, F.-Y. and Yuan, C., Bismut Formulae and Applications for Functional SPDEs, arXiv:1110.5150v1.
- Dong, Z. and Xie, Y., Ergodicity of Linear SPDE Driven by Lévy Noise, *J. Syst. Sci. Complex*, **23** (2010), 137-152.
- Elworthy, K. D. and Li, X.-M., Formulae for the derivatives of heat semigroups, *J. Funct. Anal.*, **125** (1994), 252-286.
- Guillin, A. and Wang, F.-Y., Degerate Fokker-Planck Equations: Bismut Formula, Gradient Estimate and Harnack Inequality, accessible on arXiv:1103.2817v2.

References

- Wang, F.-Y., Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds, *Ann. Probab.*, **39** (2011), 1449–1467.
- Wang, F.-Y. and Yuan, C., Harnack inequalities for functional SDEs with multiplicative noise and applications, *Stoch. Proc. Appl.*, **121** (2011), 2692–2710.
- Zhang, X., Stochastic flows and Bismut formulas for stochastic Hamiltonian systems, *Stoch. Proc. Appl.*, **120** (2010), 1929-1949.

Thanks A Lot !