Stochastic partial differential equations with reflection

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Abstract

This work is concerned with white noise driven SPDEs with reflection. The existence and uniqueness of the solution is obtained. Various properties of the solution are studied. We will discuss the strong Feller property, Harnack inequalities, Varadhan type small time asymptotics and also the large deviations. This talk is based on the following preprints:

1. T. Zhang, White noise driven SPDEs with reflection: strong Feller properties and Harnack inequalities. To appear in POTA

2. T. Xu and T. Zhang, White noise driven SPDEs with reflection: existence, uniqueness and large deviation principles. To appear in SPA.

3. J. Yang and T. Zhang, Invariant measures of stochastic partial differential equations with reflection. Preprint 2010, Manchester.

Let me first introduce the equation.

The equation

Consider the following SPDEs with reflection:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x) + \eta(t,x) \\ u(0,\cdot) = f \geq 0, \quad u(t,0) = u(t,1) = 0, \end{cases} \tag{1}$$

where $b(\cdot), \sigma(\cdot) : R \to R$ are measurable functions, $\eta(t,x)$ is a random measure which is a part of the solution pair (u, η), $\dot{W}(t,x)$ is the space-time white noise on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\mathcal{F}_t = \sigma(W(s,x), 0 \leq s \leq t, x \in [0,1])$. This equation was first studied by Nualart and Pardoux when $\sigma(\cdot) = 1$, and by Donati-Martin and Pardoux for general diffusion coefficient σ. It is an extension of the deterministic parabolic obstacle problem. SPDEs with reflection can also be used to model the evolution of random interfaces near a hard wall. It was proved by T. Funaki and S. Olla that the fluctuations of a $\nabla\phi$ interface model near a hard wall converge in law to the stationary solution of a SPDE with reflection.

The solution

A pair (u, η) is said to be a solution of equation (1) if
(i). u is a continuous processes on $R_+ \times [0, 1]$; $u(t, x)$ is $\mathcal{F}_t$ measurable and $u(t, x) \geq 0$ a.s.
(ii). η is a random measure on $R_+ \times (0, 1)$ such that
a) $\eta(\{t\} \times (0, 1)) = 0$, for $t \geq 0$;
b) $\int_0^t \int_0^1 x(1 - x)\eta(ds, dx) < \infty, t \geq 0$;
c) η is adapted in the sense that for any measurable ψ, $\int_0^t \int_0^1 \psi(s, x)\eta(ds, dx)$ is $\mathcal{F}_t$ measurable.
(iii) For any $t \in R_+$, $\phi \in C^2([0, 1])$ with $\phi(0) = \phi(1) = 0$,

$$(u(t), \phi) - \int_0^t (u(s), \phi^{''})ds - \int_0^t (b(u(s)), \phi)ds = (u(0), \phi)$$
$$+ \int_0^t \int_0^1 \phi(x)\sigma(u(s, x))W(ds, dx) + \int_0^t \int_0^1 \phi(x)\eta(ds, dx) \quad a.s., \tag{2}$$

The solution

where $(\cdot,\cdot)$ denotes the scalar product in $H = L^2([0,1])$.
(iv) $\int_0^\infty \int_0^1 u(s,x)\eta(ds,dx) = 0$.

Denote by $\mathcal{B}_b(H)$ the space of all bounded measurable functions on H. The semigroup P_t associated with the solution u is defined by

$$T_t G(f) = E[G(u(t,f))] \quad \text{for} \quad G \in \mathcal{B}_b(H),$$

where $u(t,f)$ is the solution of (1) with $u(0,f) = f \geq 0$.

Among other things we studied the strong Feller property, Harnack inequalities and the Varadhan type small time asymptotics of the semigroup $T_t, t \geq 0$. The strong Feller property of stochastic evolution equations and stochastic partial differential equations (SPDEs) was studied by many authors.

Particularly we mention the work of Peszat and Zabaczyk in Ann. Prob. in 1995. The dimension free Harnack inequality studied here was introduced by Fengyu Wang . The Varadhan type small time asymptotics for infinite dimensional diffusions has been investigated by several people. We mention the work of Hino and Ramirez in Ann. Prob. 2003, Aida and Z. in Potential Analysis in 2002. and Z. in Ann. Prob. in 2001.

For our equation (1) the main problem is to handle the reflection which introduces the extra random measure term η. This term makes it difficult to work with the equation (1) directly. To overcome this difficulty, our idea is to establish a kind of uniform strong Feller property and uniform Harnack inequality for the approximating solutions u^{ε} and then to pass to the limit. For the large deviations, we will use the weak convergence approach.

Existence and uniqueness

Here is the theorem for existence and uniqueness of the solution.

Theorem (XZ, 2008)

Suppose the functions $b : R \to R$ *and* $\sigma : R \to R$ *are Lipschitz continuous. Then the equation (1) admits a unique solution.*

Remark: Donati-Martin and Pardoux proved the existence of a minimal solution to equation (1) in 1993 (PTRF). The uniqueness was left open. We obtained the uniqueness recently and our approach also gave a much shorter proof of the existence. Next I will explain the ideas of the proof.

Deterministic obstacle problem

Consider the following deterministic parabolic obstacle problem:

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} - \frac{\partial^2 z(x,t)}{\partial x^2} = \eta(dx, dt); \\ z(x,t) \geq -v(x,t); \\ \int_0^1 \int_0^T (z(x,t) + v(x,t))\eta(dx, dt) = 0, \end{cases} \tag{3}$$

where $v \in C([0,1] \times [0,T])$ with $v(x,0) = u_0(x)$. If a pair (z, η) satisfies

(1). z is a continuous function on $[0,1] \times [0,T]$ and

$$z(x,0) = 0, z(0,t) = z(1,t) = 0, \ z \geq -v.$$

(2). η is a measure on $(0,1) \times \mathbb{R}_+$ such that for all $\varepsilon > 0$, $T > 0$

$$\eta\big((\varepsilon, (1-\varepsilon) \times [0,T])\big) < \infty.$$

(3). For all $t \geq 0, \phi \in C_0^2(0,1)$,

$$(z(t), \phi) - \int_0^t (z(s), \phi'')ds = \int_0^t \int_0^1 \phi(x)\eta(dx, ds).$$

(4). $\int_0^t \int_0^1 (z(x,s) + v(x,s))\eta(dx, ds) = 0$,
then (z, η) is called a solution to problem (3). The following result was proved in [NP]

Theorem (NP)

If $v(x,0) = u_0(x), v(0,t) = v(1,t) = 0$ for all $t \geq 0$, Eq. (3) admits a unique solution. Moreover, $|z|_\infty^T \leq |v|_\infty^T$, where $|v|_\infty^T = \sup_{0\leq x\leq 1, 0\leq t\leq T} |v(x,t)|$ and $|z|_\infty^T$ is defined similarly.

Ideas of the proof for existence and uniqueness

We will use successive iteration. Let

$$v_1(x,t) = \int_0^1 G_t(x,y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x,y)f(y,s;u_0)dyds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,s;u_0)W(dy,ds),$$

where G is the Green's function associated to the operator $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. It can be shown that $v_1(x,t)$ satisfies the following SPDE:

$$\begin{cases} \frac{\partial v_1(x,t)}{\partial t} - \frac{\partial^2 v_1(x,t)}{\partial x^2} = f(x,s;u_0) + \sigma(x,s;u_0)\dot{W}(x,t); \\ v_1(x,0) = u_0(x); \\ v_1(0,t) = v_1(1,t) = 0, \end{cases}$$

and $v_1(\cdot,\cdot) \in C([0,1]\times[0,T])$.

Ideas of the proof for existence and uniqueness

Denote by (z_1, η_1) the unique random solution of (3) with $v = v_1$. Set $u_1 = z_1 + v_1$, then we can verify that (u_1, η_1) is the unique solution of the following reflected SPDE:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} - \frac{\partial^2 u_1(x,t)}{\partial x^2} + f(x,s;u_0) = \sigma(x,s;u_0)\dot{W}(x,t) + \eta_1; \\ u_1(\cdot,0) = u_0; \\ u_1(0,t) = u_1(1,t) = 0. \end{cases} \tag{4}$$

Ideas of the proof for existence and uniqueness

Iterating the above procedure, suppose u_{n-1} has been defined. Let

$$\begin{aligned} v_n(x,t) &= \int_0^1 G_t(x,y)u_0(y)dy - \int_0^t\int_0^1 G_{t-s}(x,y)f(y,s;u_{n-1})dyds \\ &\quad + \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(y,s;u_{n-1})W(dy,ds), \end{aligned} \tag{5}$$

and (z_n, η_n) be the unique random solution of equation (3) with $v(x,t) = v_n(x,t)$. Set $u_n = z_n + v_n$, then (u_n, η_n) is the unique solution of the following reflected SPDE:

$$\begin{cases} \frac{\partial u_n(x,t)}{\partial t} - \frac{\partial^2 u_n(x,t)}{\partial x^2} + f(x,t;u_{n-1}) = \sigma(x,t;u_{n-1})\dot{W}(x,t) + \eta_n; \\ u_n(\cdot,0) = u_0; \\ u_n(0,t) = u_n(1,t) = 0. \end{cases} \tag{6}$$

Ideas of the proof for existence and uniqueness

From Theorem [NP], we have

$$|z_n - z_{n-1}|_\infty^T \le |v_n - v_{n-1}|_\infty^T,$$

hence,

$$|u_n - u_{n-1}|_\infty^T \le 2|v_n - v_{n-1}|_\infty^T. \tag{7}$$

From here we can show that

$$\begin{aligned} E(|u_n - u_{n-1}|_\infty^T)^p &\le 2^p E(|v_n - v_{n-1}|_\infty^T)^p \\ &\le c(p, K, T) E \int_0^T (|u_{n-1} - u_{n-2}|_\infty^t)^p dt \\ &\le \cdots \le c^{n-1} E(|u_1 - u_0|_\infty^T)^p \frac{T^{n-1}}{(n-1)!}, \end{aligned} \tag{8}$$

Ideas of the proof for existence and uniqueness

It follows easily now that there exists a random field $u(x,t) \in C([0,1] \times [0,T])$ such that

$$E(|u|_{\infty}^{T})^{p} < \infty, \tag{9}$$

and

$$\lim_{n\to\infty} E(|u_n - u|_{\infty}^{T})^{p} = 0.$$

We can show that u is a solution of equation (1).

Uniqueness. Let (u_1, η_1) and (u_2, η_2) be two solutions of equation (1). Set

$$\begin{aligned} v_i(x,t) &= \int_0^1 G_t(x,y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x,y)f(y,s;u_i)dyds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,s;u_i)W(dy,ds). \end{aligned} \tag{10}$$

Then $(z_i = u_i - v_i, \eta_i)$ is the unique solution of equation (3) with $v = v_i$, $i = 1, 2$. Similar to (7), we have

$$|u_1 - u_2|_\infty^T \le 2|v_1 - v_2|_\infty^T.$$

Ideas of the proof for existence and uniqueness

Using a similar argument as above, we obtain

$$E(|u_1 - u_2|_\infty^T)^p \leq 2^p E(|v_1 - v_2|_\infty^T)^p \leq cE \int_0^T (|u_1 - u_2|_\infty^t)^p dt.$$

This implies that $E|u_1 - u_2|_\infty^T = 0$ proving the uniqueness.

Invariant measures

Consider the following SPDE with two reflecting walls:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(u) + \sigma(u)\dot{W} + \eta - \xi, \tag{11}$$

under conditions

$$\begin{cases} u(0,t) = 0, \ u(1,t) = 0, \quad \text{for } t \geq 0; \\ u(x,0) = u_0(x) \in C([0,1]); \\ h^1(x) \leq u(x,t) \leq h^2(x) \quad \text{for } x \in [0,1]; \\ \int_0^T \int_0^1 \left(u(x,t) - h^1(x)\right)\eta(dx,dt) = 0; \\ \int_0^T \int_0^1 \left(h^2(x) - u(x,t)\right)\xi(dx,dt) = 0. \end{cases}$$

We assume that the reflecting walls $h^1(x)$, $h^2(x)$ are continuous functions satisfying $h^1(0) \leq 0, h^1(1) \leq 0, h^2(0) \geq 0$, and $h^2(1) \geq 0$, and

(H1) $h^1(x) < h^2(x)$ for $x \in (0,1)$;

(H2) $\frac{\partial^2 h^i}{\partial x^2} \in L^2([0,1])$, $\frac{\partial^2}{\partial x^2}$ are interpreted in a distributions sense.

Invariant measures

Using the similar method as above, we can establish the existence and uniqueness of the solutions of SPDEs with two reflecting walls. Our main interest of this section is on the invariant measures for the SPDEs with two reflecting walls. When $\sigma = 1$, it is a kind of gradient system. This case was studied by Otobe. Here are our results.

Theorem (YZ, 2010)

Assume the coefficients $b(\cdot)$ *and* $\sigma(\cdot)$ *are Lipschitz. Then there exists an invariant measure to the equation (11) on* $C([0,1])$.

Theorem (YZ, 2010)

Suppose $b(\cdot)$, $\sigma(\cdot)$ *are Lipschitz and there exits a constant* $L_0 > 0$ *such that* $\sigma(\cdot) \geq L_0$. *There is a unique invariant measure* μ *for the equation (11).*

We want to say several words about the proofs. For the existence of an invariant measure, according to Krylov–Bogoliubov theorem, if the family $P_t(u_0, \cdot),\ t > 1$ is uniformly tight, then there exists an invariant measure for (11), where $P_t(u_0, \cdot)$ denotes the transition function of the solution $u(t)$. So we need to show that for every $\varepsilon > 0$ there is a compact set $K \subset C([0,1])$ such that $\mathbb{P}(u(t) \in K) \geq 1 - \varepsilon$ for all $t > 1$. By the Markov property

$$\mathbb{P}(u(t) \in K) \ = \ \mathbb{E}\big(P_1(u(t-1), K)\big). \qquad (12)$$

Note that $u(t-1)$ is bounded between $h_1(\cdot)$ and $h_2(\cdot)$. It is enough to prove $\mathbb{P}\big(u(1, g) \in K\big) \geq 1 - \varepsilon$, for any g satisfying $h^1(x) \leq g(x) \leq h^2(x)$.

Invariant measures

Write $u(x,t,g)$ for the solution of the SPDE with reflection such that $u(x,0,g)=g(x)$. Put

$$\begin{aligned} v(x,t,g) &= \int_0^1 G_t(x,y)g(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)f(u(y,s,g))dyds \\ &\quad + \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u(y,s,g))W(dy,ds) \end{aligned} \tag{13}$$

From the relationship between u and v proved in [YZ], we have the following inequality:

$$\|u(1,g_1)-u(1,g_2)\|_\infty \le 2 \sup_{0\le t\le 1, 0\le x\le 1} |v(x,t,g_1)-v(x,t,g_2)|$$

This means that $u(\cdot,1,g)$ is a continuous functional of v, denoted by $u=\Phi(v)$. Thus, if $K_1\subset C([0,1]\times[0,1])$ is a compact set, then $K=\Phi(K_1)$ is compact in $C([0,1])$. Now for every $\varepsilon>0$, it is possible to prove that there is a compact K_1 such that

$$P(v(\cdot,\cdot,g)\in K_1)\ge 1-\varepsilon$$

Invariant measures

For the uniqueness of the invariant measures, it seems hard to use the well known method of proving the irreducibility of the SPDEs with reflection. However, we are able to adopt a coupling method used for SPDEs by Carl Muller to the SPDEs with reflection.

The approximating solutions

For each $\varepsilon > 0$, let $u^\varepsilon(t,x)$ be the solution of the penalized SPDE:

$$\left\{\begin{array}{l} \frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + b(u^\varepsilon(t,x)) \\ +\sigma(u^\varepsilon(t,x))\dot{W}(t,x) + \frac{1}{\varepsilon}(u^\varepsilon(t,x))^- \\ u^\varepsilon(0,\cdot) = f, \quad u^\varepsilon(t,0) = u^\varepsilon(t,1) = 0 \end{array}\right. \tag{14}$$

Let $H = L^2([0,1])$. The following Theorem was proved by Donati-Martin and Pardoux.

Theorem (1)

For every non-negative $f \in L^2([0,1])$, $u^\varepsilon(t,\cdot)$ converges a.s. to the solution $u(t,\cdot)$ of equation (1) in H as $\varepsilon \to 0$. Moreover, for any $p \geq 1$,

$$\sup_{\varepsilon} \sup_{t\in[0,T]} E\left[|u^\varepsilon(t,\cdot)|_H^p\right] < \infty. \tag{15}$$

Strong Feller properties

Let $e_n(x) = \sqrt{2} sin \pi nx, n \geq 1$ be the eigenvectors of the operator $A = \frac{\partial^2}{\partial x^2}$ constituting an orthonormal system of H. Put

$$\beta_n(t) = \int_0^t \int_0^1 e_n(x) W(ds, dx).$$

$\beta_n(t), n \geq 1$ is a sequence of independent Brownian motions.
Define a mapping $\Sigma(\cdot) : H \rightarrow L(H)$ by

$$\Sigma(f)h = \sigma(f(x))h(x) \quad f, h \in H,$$

and a H-cylindrical Brownian motion $W(t)$ by

$$W(t) = \sum_{n=1}^{\infty} \beta_n(t) e_n.$$

Then

$$\sigma(u(t, x))\dot{W}(t, x) = \Sigma(u(t))dW(t)$$

is the stochastic Itô integral against the cylindrical Browninan motion.

Strong Feller properties

Theorem (2)

Assume $b(\xi)$, $\sigma(\xi)$ *are Lipschitz continuous and there exist positive constants* k_1, k_2 *such that* $k_1 \leq |\sigma(\xi)| \leq k_2$. *Then for any* $T > 0$ *there exists a constant* C_T *such that for all* $G \in \mathcal{B}_b(H)$ *and* $t \in (0, T]$

$$|T_t G(f_1) - T_t G(f_2)| \leq \frac{C_T}{\sqrt{t}} ||G||_\infty |f_1 - f_2|_H, \tag{16}$$

for $f_1, f_2 \in H$ *with* $f_1 \geq 0$, $f_2 \geq 0$, *where* $||G||_\infty = \sup_f |G(f)|$. *In particular,* $T_t, t \geq 0$ *is strong Feller.*

The proof

In view of Theorem 1, to prove (16) it is enough to show that there exists a constant C_T, independent of ε, such that

$$|T_t^\varepsilon G(f_1) - T_t^\varepsilon G(f_2)| \leq \frac{C_T}{\sqrt{t}} ||G||_\infty |f_1 - f_2|_H, \quad f_1, f_2 \in H. \quad (17)$$

Take a non-negative function $\phi \in C_0^\infty(R)$ with $\int_R \phi(x)dx = 1$. Put

$$b_n(\xi) = n \int_R \phi(n(\xi - y))b(y)dy,$$

$$\sigma_n(\xi) = n \int_R \phi(n(\xi - y))\sigma(y)dy,$$

$$h_n(\xi) = n \int_R \phi(n(\xi - y))y^- dy.$$

Let $u_n^\varepsilon(t, x)$ be the solution of the following SPDE:

The proof

$$\begin{cases} \frac{\partial u_n^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 u_n^\varepsilon(t,x)}{\partial x^2} + b_n(u_n^\varepsilon(t,x)) \\ +\sigma_n(u_n^\varepsilon(t,x))\dot{W}(t,x) + \frac{1}{\varepsilon}h_n(u_n^\varepsilon(t,x)), \\ u_n^\varepsilon(0,\cdot) = f, \quad u_n^\varepsilon(t,0) = u_n^\varepsilon(t,1) = 0. \end{cases} \tag{18}$$

One can verify that for any fixed $\varepsilon > 0$ and $p \geq 1$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} E[|u_n^\varepsilon(t,\cdot) - u^\varepsilon(t,\cdot)|_H^p] = 0. \tag{19}$$

For $G \in C_b^2(H)$, define $T_t^{n,\varepsilon}G(f) = E[G(u_n^\varepsilon(t,f))]$. Because of (19), the statement (17) will follow if we can show that there exists a constant C_T, independent of ε and n, such that

$$|T_t^{n,\varepsilon}G(f_1) - T_t^{n,\varepsilon}G(f_2)| \leq \frac{C_T}{\sqrt{t}}||G||_\infty|f_1 - f_2|_H, \quad f_1, f_2 \in H. \tag{20}$$

The proof

Denote by $Du_n^\varepsilon(t,f,x)(g)$ the directional derivative of $u_n^\varepsilon(t,f,\cdot)$ at f in the direction g, then it satisfies the following mild form of the linear SPDE:

$$\begin{aligned}
& Du_n^\varepsilon(t,f,x)(g) \\
= \; & P_t g(x) + \int_0^t \int_0^1 P_{t-s}(x,y) b_n'(u_n^\varepsilon(s,f,y)) Du_n^\varepsilon(s,f,y)(g) ds dy \\
+ \; & \frac{1}{\varepsilon} \int_0^t P_{t-s}(x,y) h_n'(u_n^\varepsilon(s,f,y)) Du_n^\varepsilon(s,f,y)(g) ds dy \\
+ \; & \int_0^t P_{t-s}(x,y) \sigma_n'(u_n^\varepsilon(s,f,y)) Du_n^\varepsilon(s,f,y)(g) W(ds,dy). \quad (21)
\end{aligned}$$

Assume first $g(x) \geq 0, a.e.$ Since $h_n'(\xi) \leq 0$, by the comparison theorem of SPDEs we deduce that

$$0 \leq Du_n^\varepsilon(t,f,x)(g) \leq Y_n^\varepsilon(t,f,g,x), \quad (22)$$

The proof

where $Y_n^\varepsilon(t,f,g,x), t \geq 0$ is the solution of the following linear SPDE:

$$\begin{aligned} & Y_n^\varepsilon(t,f,g,x) \\ = \; & P_t g(x) + \int_0^t \int_0^1 P_{t-s}(x,y) b_n'(u_n^\varepsilon(s,f,y)) Y_n^\varepsilon(s,f,g,y) ds dy \\ + \; & \int_0^t P_{t-s}(x,y) \sigma_n'(u_n^\varepsilon(s,f,y)) Y_n^\varepsilon(s,f,g,y) W(ds,dy). \end{aligned} \tag{23}$$

By virtue of Burkhölder inequality and Grownwall type inequality one can show that

$$\begin{aligned} & \sup_{\varepsilon>0, t\in[0,T]} E[| Y_n^\varepsilon(t,f,g,\cdot)|_H^2] \\ = & \sup_{\varepsilon>0, t\in[0,T]} E\left[\int_0^1 (Y_n^\varepsilon(t,f,g,y))^2 dy \right] \leq C_1 |g|_H^2. \end{aligned}$$

(22) implies that the same is true for $Du_n^\varepsilon(t,f,x)(g)$, i.e.,

The proof

$$\sup_{\varepsilon>0, t\in[0,T]} E\left[\int_0^1 (Du_n^\varepsilon(t,f,y)(g))^2 dy\right] \le C_1 |g|_H^2. \qquad (24)$$

The proof

For a general $g \in H$, writing $g(x) = g^+(x) - g^-(x)$ we have

$$Du_n^\varepsilon(t,f,x)(g) = Du_n^\varepsilon(t,f,x)(g^+) - Du_n^\varepsilon(t,f,x)(g^-).$$

It follows from (24) that

$$\sup_{\varepsilon>0, t\in[0,T]} E\left[\int_0^1 (Du_n^\varepsilon(t,f,y)(g))^2 dy\right] \leq 2C_1|g|_H^2. \quad g \in H.$$

Let $G \in C_b^2(H)$. By the Elworthy formula (Lemma 7.1.3 in [DZ]), we have

$$\begin{aligned} &\left\langle DT_t^{n,\varepsilon}G(f), g\right\rangle \\ = \ & \frac{1}{t}E\left\{G(u_n^\varepsilon(t,f))\int_0^t \langle(\Sigma_n)^{-1}(u_n^\varepsilon(s,f))Du_n^\varepsilon(s,f)(g), W(ds)\rangle\right\} \end{aligned} \tag{25}$$

The theorem follows.

Harnack inequalities

Consider the SPDE with reflection driven by additive noise:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(u(t,x)) + \dot{W}(t,x) + \eta(t,x) \\ u(0,\cdot) = f \geq 0, \quad u(t,0) = u(t,1) = 0. \end{cases} \tag{26}$$

As before, $T_t, t \geq 0$ denotes the semigroup generated by the solution. Here is the Harnack inequality:

Theorem (3)

Assume that there exists a constant c such that

$$(\xi_1 - \xi_2)(b(\xi_1) - b(\xi_2)) \leq c|\xi_1 - \xi_2|^2 \tag{27}$$

for any $\xi_1, \xi_2 \in R$. Then for any $t > 0$, any non-negative bounded measurable function G on H, any $\alpha > 1$ and any $f_1 \in H, f_2 \in H$ with $f \geq 0$, $f_2 \geq 0$,

$$(T_t G)^\alpha(f_1) \leq T_t(G^\alpha)(f_2) exp\left(\frac{\alpha}{\alpha - 1}(c^2 t + \frac{1}{t})|f_1 - f_2|_H^2\right). \tag{28}$$

Sketch of the Proof

We will adopt the coupling method. Recall the following penalized SPDE:

$$\begin{cases} \frac{\partial u^\varepsilon(t,f,x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t,f,x)}{\partial x^2} + b(u^\varepsilon(t,f,x)) \\ +\dot{W}(t,x) + \frac{1}{\varepsilon}(u^\varepsilon(t,f,x))^- \\ u^\varepsilon(0,f,\cdot) = f, \quad u^\varepsilon(t,f,0) = u^\varepsilon(t,f,1) = 0. \end{cases} \tag{29}$$

It is sufficient to show

$$(T_t^\varepsilon G)^\alpha(f_1) \leq T_t^\varepsilon(G^\alpha)(f_2) exp\left(\frac{\alpha}{\alpha-1}(c^2 t + \frac{1}{t})|f_1 - f_2|_H^2\right), \tag{30}$$

where $T_t^\varepsilon, t \geq 0$ denotes the semigroup associated with u^ε. Suppose $f_1 \in H, f_2 \in H$ with $f_1 \neq f_2$. With $A = \frac{\partial^2}{\partial x^2}$ we have

$$\begin{aligned} u^\varepsilon(r,f_1) &= f_1 + \int_0^r Au^\varepsilon(s,f_1)ds + \int_0^r b(u^\varepsilon(s,f_1))ds \\ &\quad + \int_0^r \frac{1}{\varepsilon}(u^\varepsilon(s,f_1))^- ds + W(r). \end{aligned} \tag{31}$$

Sketch of the proof

Let v^ε be the solution of the following equation:

$$\begin{aligned} & v^\varepsilon(r, f_2) \\ = \quad & f_2 + \int_0^r Av^\varepsilon(s, f_2)ds + \int_0^r b(v^\varepsilon(s, f_2))ds \\ & + \int_0^r \frac{1}{\varepsilon}(v^\varepsilon(s, f_2))^- ds + W(r) + \int_0^r (c(u^\varepsilon(s, f_1) - v^\varepsilon(s, f_2))I_{\{s<\tau\}}ds \\ & + \int_0^r \left(\frac{u^\varepsilon(s, f_1) - v^\varepsilon(s, f_2)}{|u^\varepsilon(s, f_1) - v^\varepsilon(s, f_2)|_H} \frac{|f_1 - f_2|_H}{t} \right) I_{\{s<\tau\}}ds, \qquad (32) \end{aligned}$$

where $\tau := \inf\{s \geq 0 : u^\varepsilon(s, f_1) = v^\varepsilon(s, f_2)\}$. By the chain rule and the fact that the function y^- is decreasing and the operator A is negative, we can show that

$$|u^\varepsilon(r \wedge \tau, f_1) - v^\varepsilon(r \wedge \tau, f_2)|_H \leq |f_1 - f_2|_H - \frac{r \wedge \tau}{t}|f_1 - f_2|_H. \quad (33)$$

Sketch of the proof

This implies that $\tau \leq t$ and $u^\varepsilon(t, f_1) = v^\varepsilon(t, f_2)$. Otherwise, if $t < \tau$, we deduce from (33) that $u^\varepsilon(t, f_1) = v^\varepsilon(t, f_2)$ which contradicts the definition of the stopping time τ. Put

$$\xi_s = \left(c(u^\varepsilon(s, f_1) - v^\varepsilon(s, f_2)) + \frac{u^\varepsilon(s, f_1) - v^\varepsilon(s, f_2)}{|u^\varepsilon(s, f_1) - v^\varepsilon(s, f_2)|_H} \frac{|f_1 - f_2|_H}{t} \right) I_{\{s<\tau\}}.$$

Then,

$$\int_0^t |\xi_s|_H^2 ds \leq 2(c^2 t + \frac{1}{t})|f_1 - f_2|_H^2. \tag{34}$$

Sketch of the proof

Set

$$Z_t = exp\{-\int_0^t < \xi_s, dW_s > -\frac{1}{2}\int_0^t |\xi_s|_H^2 ds\},$$

and define a new probability measure Q as

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t.$$

By the Girsanov Theorem, $\tilde{W}(u) = W(u) + \int_0^u \xi_s ds, u \geq 0$ is a H-cylindrical Brownian motion under Q. Since

$$\begin{aligned} v^\varepsilon(r, f_2) &= f_2 + \int_0^r Av^\varepsilon(s, f_2)ds + \int_0^r b(v^\varepsilon(s, f_2))ds \\ &\quad + \int_0^r \frac{1}{\varepsilon}(v^\varepsilon(s, f_2))^- ds + \tilde{W}(r), \end{aligned} \tag{35}$$

we see that the law of $v^\varepsilon(r, f_2)$ under Q is the same as that of $u^\varepsilon(r, f_2)$ under P.

Sketch of the proof

For $\alpha > 1$, we have

$$\begin{aligned} T_t^\varepsilon G(f_1) &= E[G(u^\varepsilon(t,f_1))] = E[G(v^\varepsilon(t,f_2))] \\ &= E[G(v^\varepsilon(t,f_2))Z_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}}] \\ &\leq (E[G^\alpha(v^\varepsilon(t,f_2))Z_t])^{\frac{1}{\alpha}} (E[Z_t^{-\frac{1}{\alpha-1}}])^{\frac{\alpha-1}{\alpha}} \\ &= (E[G^\alpha(u^\varepsilon(t,f_2))])^{\frac{1}{\alpha}} (E[Z_t^{-\frac{1}{\alpha-1}}])^{\frac{\alpha-1}{\alpha}} \\ &= (T_t^\varepsilon(G^\alpha)(f_2))^{\frac{1}{\alpha}} (E[Z_t^{-\frac{1}{\alpha-1}}])^{\frac{\alpha-1}{\alpha}}. \end{aligned} \tag{36}$$

The exponential martingale can be bounded as follows:

$$\begin{aligned} &(E[Z_t^{-\frac{1}{\alpha-1}}])^{\frac{\alpha-1}{\alpha}} \\ \leq\ & exp\left(\frac{1}{\alpha-1}(c^2 t + \frac{1}{t})|f_1 - f_2|_H^2\right), \end{aligned} \tag{37}$$

Combining (37) with (36) yields

$$(T_t^\varepsilon G(f_1))^\alpha \leq T_t^\varepsilon(G^\alpha)(f_2) exp\left(\frac{\alpha}{\alpha-1}(c^2 t + \frac{1}{t})|f_1 - f_2|_H^2\right).$$

Corollary (4)

The semigroup $T_t, t \geq 0$ is strong Feller, i.e., $T_t G(f)$ is a continuous function on H for every $G \in \mathcal{B}_b(H)$.

From now on, we assume $b = 0$ in (26). Namely, consider

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \dot{W}(t,x) + \eta(t,x) \\ u(0,\cdot) = f \geq 0, \quad u(t,0) = u(t,1) = 0 \end{cases} \tag{38}$$

It was proved by Zambotti that the solution $u(t), t \geq 0$ to equation (38) is symmetric with respect to the law $\nu(dz)$ of the $3 - D$ Bessel bridge and the associated Dirichlet form is given by

$$\begin{cases} \mathcal{E}(F,G) = \int_{K_0} < \nabla F, \nabla G >_H (z)\nu(dz), \\ D(\mathcal{E}) \supset \{\text{Lipschitz continuous functions}\}, \end{cases} \tag{39}$$

where $K_0 = \{h \in H; h \geq 0\}$, $\nabla F, \nabla G$ denote the gradient of F and G in H.

Now we can state the second corollary:

Corollary (5)

The semigroup $T_t, t \geq 0$ admits the hyperbound property, that is, T_t is a bounded linear operator from $L^2(\nu)$ into $L^4(\nu)$ for sufficiently large $t > 0$.

Small time asymptotics

For $B \subset K_0, C \subset K_0$ with $\nu(B) > 0, \nu(C) > 0$, define

$$\begin{aligned} & d(B, C) \\ = \; & sup\{essinf_{f \in B} essinf_{g \in C}(|f - g|_H), essinf_{g \in C} essinf_{f \in B}(|f - g|_H)\}, \end{aligned} \tag{40}$$

where *esinf* is taken with respect to the symmetrizing measure $\nu(dz)$. Let P_ν be the law of the process $u(t), t \geq 0$ with the initial distribution equal to ν. As an another application of the Harnack inequality, we have the following Varadhan type small type asymptotics:

Theorem (6)

Suppose $B \subset K_0, C \subset K_0$ with $\nu(B) > 0, \nu(C) > 0$. Then

$$\lim_{t \to 0} tlog P_\nu(u(t) \in B, u(0) \in C) = -d^2(B, C). \tag{41}$$

Sketch of the proof

Upper bound. Take $\lambda < d(B, C)$. Then one of the terms on the right of (40) will be bigger than λ, say

$$essinf_{f \in B} essinf_{g \in C}(|f - g|_H) > \lambda.$$

This implies that there exists $B_1 \subset B$ with $\nu(B \setminus B_1) = 0$ such that

$$essinf_{g \in C}(|f - g|_H) > \lambda \qquad \text{for} \quad f \in B_1. \tag{42}$$

Set

$$F(f) := essinf_{g \in C}(|f - g|_H).$$

We can show that

$$|F(f_1) - F(f_2) \leq |f_1 - f_2|_H.$$

Small time asymptotics

As a consequence, $F \in D(\mathcal{E})$ and $|\nabla F|_H \leq 1$. Let $\tilde{C}$ denote the support of the measure $I_C(z)\nu(dz)$. It is easy to see that $F(f) = 0$ on $\tilde{C}$. This together with (42) yields

$$P_\nu(u(t) \in B, u(0) \in C) = P_\nu(u(t) \in B_1, u(0) \in \tilde{C})$$

$$\leq P_\nu(F(u(t)) > \lambda, , F(u(0)) = 0)$$

$$\leq P_\nu(F(u(t)) - F(u(0)) > \lambda)$$

From here, using Lyons-Zheng decomposition we obtain that

$$\lim_{t \to 0} tlogP_\nu(u(t) \in B, u(0) \in C) \leq -\lambda^2. \tag{43}$$

Letting $\lambda \to d(B, C)$ we get the upper bound.

Small time asymptotics

Lower bound. For the lower bound, we may assume $d(B,C)<\infty$. Take any $\eta > d(B,C)$. We can find $B_1 \subset B$ with $\nu(B_1)>0$ such that

$$essinf_{g\in C}(|f-g|_H) < \eta, \qquad \text{for} \quad f \in B_1.$$

This implies that for every $f \in B_1$, there is further a subset $C_f \subset C$ with $\nu(C_f)>0$ such that

$$|f-g|_H < \eta, \qquad \text{for} \quad g \in C_f. \tag{44}$$

Since $\nu(B_1)>0$ and $T_t(I_{B_1}) \to I_{B_1}$ in $L^2(\nu)$ as $t \to 0$, we can find $f_0 \in B_1$ such that $T_t(I_{B_1})(f_0) \to I_{B_1}(f_0) = 1$ as $t \to 0$ (taking a subsequence if necessary).

Small time asymptotics

Now by the Harnack inequalities, for any $\alpha > 1$, we have

$$\begin{aligned}
P_\nu(u(t) \in B, u(0) \in C) &= \int_{K_0} T_t I_B(z) I_C(z)\nu(dz) \\
&\geq \int_{K_0} T_t I_{B_1}(z) I_C(z)\nu(dz) = \int_{K_0} T_t I^{\alpha}_{B_1}(z) I_C(z)\nu(dz) \\
&\geq \int_{K_0} (T_t I_{B_1}(f_0))^{\alpha} exp(-\frac{\alpha}{\alpha-1}\frac{1}{t}|f_0 - z|_H^2) I_C(z)\nu(dz) \\
&\geq \int_{K_0} (T_t I_{B_1}(f_0))^{\alpha} exp(-\frac{\alpha}{\alpha-1}\frac{1}{t}|f_0 - z|_H^2) I_{C_{f_0}}(z)\nu(dz) \\
&\geq (T_t I_{B_1}(f_0))^{\alpha} exp(-\frac{\alpha}{\alpha-1}\frac{1}{t}\eta^2)\nu(C_{f_0}),
\end{aligned} \tag{45}$$

where (33) has been used for $f = f_0$.

Small time asymptotics

With the special choice of f_0, it follows from (34) that

$$\begin{aligned}
& \lim_{t\to 0} tlogP_\nu(u(t)\in B, u(0)\in C) \\
\geq & \lim_{t\to 0} tlog(T_t I_{B_1}(f_0))^\alpha + \lim_{t\to 0} tlog\nu(C_{f_0}) - \frac{\alpha}{\alpha-1}\eta^2 \\
= & -\frac{\alpha}{\alpha-1}\eta^2 \qquad (46)
\end{aligned}$$

Let $\alpha \to \infty$ to obtain

$$\lim_{t\to 0} tlogP_\nu(u(t)\in B, u(0)\in C) \geq -\eta^2.$$

Sending η to $d(B,C)$ we reach the upper bound:

$$\lim_{t\to 0} tlogP_\nu(u(t)\in B, u(0)\in C) \geq -d^2(B,C).$$

The Cameron-Martin space associated with the Brownian sheet $\{W(x,t), x \in [0,1], t \in \mathbb{R}_+\}$ is given by

$$\mathcal{H} = \{h = \int_0^{\cdot} \int_0^{\cdot} \dot{h}(x,s)dxds; \int_0^T \int_0^1 \dot{h}^2(x,s)dxds < \infty\}.$$

For $h = \int_0^{\cdot} \int_0^{\cdot} \dot{h}(x,s)dxds \in \mathcal{H}$, consider the following reflected deterministic PDE (the skeleton equation):

$$\begin{cases} \frac{\partial s^h(x,t)}{\partial t} - \frac{\partial^2 s^h(x,t)}{\partial x^2} + f(x,t;s^h) = \sigma(x,t;s^h)\dot{h}(x,t) + \eta^h; \\ s^h(\cdot,0) = u_0; \\ s^h(0,t) = s^h(1,t) = 0. \end{cases} \quad (47)$$

As in the SPDE case, equation (44) has a unique solution.

Large deviations: the rate function

Define a function by

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathcal{H}, s^h(\cdot,\cdot)=f\}} |h|^2_{\mathcal{H}}, \quad f \in C_+([0,1] \times [0,T])$$

with the convention $\inf\{\emptyset\} = \infty$.

Theorem (7)

The function $I(\cdot)$ defined above is a good rate function on $C_+([0,1] \times [0,T])$, that is, $\{f : I(f) \leq a\}$ is compact for any $a \geq 0$.

Small noise perturbation

Consider the following small noise perturbation of the SPDEs with reflection:

$$\begin{cases} \frac{\partial u^\varepsilon(x,t)}{\partial t} - \frac{\partial^2 u^\varepsilon(x,t)}{\partial x^2} + f(x,t;u^\varepsilon) = \varepsilon\sigma(x,t;u^\varepsilon)\dot{W}(x,t) + \eta^\varepsilon; \\ u^\varepsilon(\cdot,0) = u_0; \\ u^\varepsilon(0,t) = u^\varepsilon(1,t) = 0. \end{cases} \tag{48}$$

Recall that u^ε denotes the solution of Eq(48). Here is the large deviation principle:

Theorem (8)

The laws μ_ε of $\{u^\varepsilon(\cdot,\cdot)\}_{\varepsilon>0}$ satisfy a large deviation principle on $C_+([0,1]\times[0,T])$ with the rate function $I(\cdot)$, i.e.,
(i) For any closed subset $C\subset C_+([0,1]\times[0,T])$,

$$\limsup_{\varepsilon\to 0}\varepsilon^2\log\mu_\varepsilon(C)\leq -\inf_{f\in C} I(f).$$

(ii) For any open set $G\subset C_+([0,1]\times[0,T])$,

$$\liminf_{\varepsilon\to 0}\varepsilon^2\log\mu_\varepsilon(G)\geq -\inf_{f\in G} I(f).$$

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