Stability in Distribution of SPDEs

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Outline

Ornstein-Uhlenbeck process

$$dx(t) = -\alpha x(t)dt + \sigma dB(t) \ \text{ on } t \geq 0$$

with initial value $x(0) = x_0$. The unique solution is

$$x(t) = e^{-\alpha t}x_0 + \sigma \int_0^t e^{-\alpha(t-s)}dB(s).$$

It has the mean

$$Ex(t) = e^{-\alpha t}x_0$$

and the variance

$$\mathrm{Var}(x(t)) = E|x(t) - Ex(t)|^2 = e^{-2\alpha t}x_0 + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

The distribution of $x(t)$ will converge to the normal distribution $N(0, \sigma^2/2\alpha)$.

For SDEs ($\mathbb{R}^n$)

- Arnold, Bhattacharya, Friedman, Hasminskii, Meyn, Tweedie, . . .
- Basak, Bisi, Ghosh (1996).
- Mattingly, Stuart, Higham (2002).
- Yuan & Mao (2003) and Xi (2004).

For infinite dimensional systems

- Da Prato & Zabczyk, Rockner, Zhang. . .

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- Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P}\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.
- Let H, K be two real separable Hilbert spaces. Denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from K into H.
- Denote by $\{W(t), t \geq 0\}$ a K-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Wiener process with covariance operator Q, i.e.,

$$E\langle W(t), x\rangle_K \langle W(s), y\rangle_K = (t \wedge s)\langle Qx, y\rangle_K, \quad \forall x, y \in K,$$

where Q is a positive, self-adjoint, trace class operator on K.

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- Let $K_0 = Q^{\frac{1}{2}}(K)$ be the subspace of K.
- Let $\mathcal{L}_2^0 = \mathcal{L}_2(K_0, H)$ denote the space of all Hilbert-Schmidt operators from K_0 into H, equipped with the norm

$$\|\Phi\|_{\mathcal{L}_2^0}^2 = tr((\Phi Q^{\frac{1}{2}})(\Phi Q^{\frac{1}{2}})^*) \quad \textit{for any } \Phi \in \mathcal{L}_2^0.$$

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$$\|\Phi\|^2_{\mathcal{L}_2^0} = tr((\Phi Q^{\frac{1}{2}})(\Phi Q^{\frac{1}{2}})^*) \quad \textit{for any } \Phi \in \mathcal{L}_2^0.$$

In this talk, we consider the following semi-linear stochastic partial differential equation

$$dX(t) = [AX(t) + F(X(t))]dt + G(X(t))dW(t), \quad t \geq 0, \quad (1.1)$$

with initial data $X(0) = \xi \in H$.

- A, generally unbounded, is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$, of contraction.
- The mappings $F : H \to H$, $G : H \to \mathcal{L}(K, H)$ are both Borel measurable and satisfy the following Lipschitz condition

$$\|F(x) - F(y)\|_H + \|G(x) - G(y)\|_{\mathcal{L}_2^0} \leq L\|x - y\|_H,$$

for some constant $L > 0$ and arbitrary $x, y \in H$.

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for some constant $L > 0$ and arbitrary $x, y \in H$.

Definition

A stochastic process $\{X(t), t \in [0, T]\}, 0 \leq T < \infty$, is called a strong solution of (1.1) if
(i) $X(t)$ is adapted to $\mathcal{F}_t$ and continuous in t wp 1;
(ii) $X(t) \in \mathcal{D}(A)$, the domain of A, on $[0, T] \times \Omega$ with $\int_0^T \|AX(t)\|_H dt < \infty$ with probability one,

$$X(t) = \xi + \int_0^t [AX(s) + F(X(s))]ds + \int_0^t G(X(s))dW(s)$$

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(ii) $X(t)$ is measurable and $\int_0^T \|X(t)\|_H^2 ds < \infty$ wp 1,

$$X(t) = T(t)\xi + \int_0^t T(t-s)F(X(s))ds + \int_0^t T(t-s)G(X(s))dW(s)$$

for all $t \in [0, T]$ with probability one.

Definition

The process $X(t)$ with the initial state $y(0) = \xi$ is said to be stable in distribution if there exists a probability measure $\pi(\cdot)$ on H such that $P(X^{\xi}(t) \in dy)$ converges weakly to $\pi(dy)$ as $t \to \infty$ for any $\xi \in H$. (1.1) is said to be stable in distribution if $X(t)$ is stable in distribution.

Remark Since the mild solution $X(t)$ to (1.1) is a strong Markov process, using Kolmogorov-Chapman equation, it is not difficult to show that the stability in distribution of mild solution $X(t)$ implies the existence of a unique invariant probability measure for mild solution $X(t)$.

Lemma

Suppose $V \in C^2(H; R+)$ *and* $\{X(t), t \geq 0\}$ *is a strong solution of* (1.1), *for* $t \geq 0$

$$V(X(t)) = V(\xi) + \int_0^t \mathcal{L}V(X(s))ds + \int_0^t \langle V_x(X(s)), G(X(s))dW(s)\rangle_H,$$

where, $\forall x \in \mathcal{D}(A)$

$$\mathcal{L}V(x) = \langle V_x(x), Ax + F(x)\rangle_H + \frac{1}{2}tr(V_{xx}(x)G(x)QG^*(x)).$$

we introduce the following approximating system of (1.1), for $t \geq 0$

$$\begin{aligned} dX_n(t) =& AX_n(t)dt + R(n)F(X_n(t))dt + R(n)G(X_n(t))dW(t), \\ X(0) =& R(n)\xi \in \mathcal{D}(A), \end{aligned} \tag{1.2}$$

where $n \in \rho(A)$, the resolvent set of A and $R(n) = nR(n, A)$, $R(n, A)$ is the resolvent of A.

Similar to operator $\mathcal{L}$ defined in Lemma 4, the operator $\mathcal{L}_n$ associated with (1.2), for any $x \in \mathcal{D}(A)$, can be defined by

$$\begin{aligned}\mathcal{L}_n V(x) = &\langle V_x(x), Ax + R(n)F(x)\rangle_H \\ &+ \frac{1}{2} tr(V_{xx}(x))R(n)G(x)Q(R(n)G(x))^*).\end{aligned}$$

Lemma

Under the condition $(H1)$ *and* $(H2)$, (1.2) *has a unique strong solution* $X_n(t)$ *which lies in* $C(0, T; L^2(\Omega, \mathcal{F}, P; H))$ *for all* $T \geq 0$. *Moreover,* $X_n(t)$ *converges to the mild solution* $X(t)$ *of* (1.1) *in* $C(0, T; L^2(\Omega, \mathcal{F}, P; H))$ *as* $n \to \infty$.

Lemma

Let conditions (H1) and (H2) hold. Assume that there exists a function $V(x) \in C^2(H; R_+)$ *such that for any* $x \in H$

$$\begin{aligned} c\|x\|_H^2 &\geq V(x) + \|x\|_H \|V_x(x)\|_H + \|x\|_H^2 \|V_{xx}(x)\|, \\ c_1\|x\|_H^2 &\leq V(x), \end{aligned} \tag{2.1}$$

Moreover, assume

$$\mathcal{L}V(x) \leq -\lambda_1 V(x) + \beta \quad x \in \mathcal{D}(A) \tag{2.2}$$

Then, for any $\xi \in H$ *and* $\epsilon > 0$*, there exists a constant* $M > 0$ *such that for any* $t \geq 0$

$$P\{\|X(t)\|_H \geq M\} < \epsilon. \tag{2.3}$$

In what follows we need to consider the difference between two mild solutions of (1.1) starting from different initial data, namely for any $t \geq 0$

$$X^{\xi}(t) - X^{\eta}(t) = T(t)\xi - T(t)\eta + \int_0^t T(t-s)[F(X^{\xi}(s)) - F(X^{\eta}(s))]ds + \int_0^t T(t-s)[G(X^{\xi}(s)) - G(X^{\eta}(s))]dW(s). \tag{2.4}$$

Now, for $t \geq 0$ we introduce an approximating system in correspondence with (2.4)

$$\begin{aligned} d[X_n^\xi(t) - X_n^\eta(t)] =& A[X_n^\xi(t) - X_n^\eta(t)]dt + R(n)[F(X_n^\xi(t)) - F(X_n^\eta(t))]dt \\ &+ R(n)[G(X_n^\xi(t)) - G(X_n^\eta(t))]dW(t), \\ X_n^\xi(0) - X_n^\eta(0) =& R(n)(\xi - \eta) \in \mathcal{D}(A), \end{aligned} \tag{2.5}$$

where $n \in \rho(A)$, the resolvent set of A and $R(n) = nR(n, A)$, $R(n, A)$ is the resolvent of A.

For given $U \in C^2(H; R+)$, define an operator $\mathcal{L}_n U : H \times H \to R$ associated with (2.5) by for any $x, y \in \mathcal{D}(A)$

$$\mathcal{L}_n U(x,y) = \langle U_x(x-y), A(x-y) + R(n)(F(x) - F(y))\rangle_H$$
$$+\frac{1}{2} tr(U_{xx}(x-y)R(n)(G(x) - G(y))Q(R(n)(G(x) - G(y)))^*).$$

Lemma

Let conditions $(H1)$ *and* $(H2)$ *hold. For any* $x \in H$ *assume that there exists function* $U(x) \in C^2(H; R+)$ *such that with some constants* $d, c_2, \lambda_2 \geq 0$

$$
\begin{aligned}
d\|x\|_H^2 &\geq U(x) + \|x\|_H \|U_x(x)\|_H + \|x\|_H^2 \|U_{xx}(x)\|, \\
c_2\|x\|_H^2 &\leq U(x).
\end{aligned}
\tag{2.6}
$$

$$
\mathcal{L}U(x, y) \leq -\lambda_2 U(x - y). \tag{2.7}
$$

Then, for any $\epsilon > 0$ *and any compact subset* $\mathcal{K}$ *of* H*, there exists a* $T = T(\epsilon, \mathcal{K}) > 0$ *such that*

$$
P\{\|X^\xi(t) - X^\eta(t)\|_H < \epsilon\} \geq 1 - \epsilon, \qquad t \geq T \tag{2.8}
$$

whenever $\xi, \eta \in \mathcal{K}$.

Sketch of Proof

It is easy to see from (2.6) that $U(0) = 0$. For any $\epsilon \in (0,1)$, by the continuity of U, we then can choose $\alpha \in (0, \epsilon)$ sufficiently small such that

$$\frac{\sup_{\|x\|_H \le \alpha} U(x)}{c_2 \epsilon^2} < \frac{\epsilon}{2}. \tag{2.9}$$

Denote by $X^\xi(t)$ and $X^\eta(t)$ two different mild solutions to (1.1) starting from initial datums ξ and η, respectively. Let $\mathcal{K}$ be any compact subset of H and fix any $\xi, \eta \in \mathcal{K}$. For $\beta > \alpha$, we define two stopping times as follows:

$$\tau_\alpha = \inf\{t \ge 0 : \|X^\xi(t) - X^\eta(t)\|_H \le \alpha\},$$
$$\tau_\beta = \inf\{t \ge 0 : \|X^\xi(t) - X^\eta(t)\|_H \ge \beta\}.$$

Sketch of Proof

Set $t_\beta = \tau_\beta \wedge t$. Using the Itô formula (i.e. Lemma 4) to function $U(x)$ and strong solution $X_n^\xi(t) - X_n^\eta(t)$ to (2.5),

$$
\begin{aligned}
&\mathbb{E}U(X_n^\xi(t_\beta) - X_n^\eta(t_\beta)) \\
&= \mathbb{E}U(R(n)(\xi-\eta)) + \mathbb{E}\int_0^{t_\beta} \mathcal{L}_n U(X_n^\xi(s), X_n^\eta(s))ds \\
&= \mathbb{E}U(R(n)(\xi-\eta)) + \mathbb{E}\int_0^{t_\beta} \mathcal{L} U(X_n^\xi(s), X_n^\eta(s))ds \\
&+ \mathbb{E}\int_0^{t_\beta} [\mathcal{L}_n U(X_n^\xi(s), X_n^\eta(s)) - \mathcal{L} U(X_n^\xi(s), X_n^\eta(s))]ds.
\end{aligned}
$$

Sketch of Proof

$$\mathbb{E}U(X^{\xi}(t_{\beta}) - X^{\eta}(t_{\beta})) \leq \mathbb{E}U(\xi - \eta) - \lambda_2 \mathbb{E}\int_0^{t_{\beta}} U(X^{\xi}(s) - X^{\eta}(s))ds. \quad (2.10)$$

By (2.6), it directly follows that

$$c_2 \mathbb{E}[\|X^{\xi}(\tau_{\beta}) - X^{\eta}(\tau_{\beta})\|_H^2 I_{\{\tau_{\beta} \leq t\}}] \leq \mathbb{E}U(\xi - \eta),$$

which, together with the definition of τ_{β}, gives that

$$P\{\tau_{\beta} \leq t\} \leq \frac{\mathbb{E}U(\xi - \eta)}{c_2 \beta^2}.$$

Hence, there exists a $\beta = \beta(\mathcal{K}, \epsilon) > 0$ such that

$$P\{\tau_{\beta} < \infty\} \leq \frac{\epsilon}{4}. \quad (2.11)$$

Sketch of Proof

Fix the β and let $t_\alpha = \tau_\alpha \wedge \tau_\beta \wedge t$. In the same way as (2.10) was done, we can obtain from (2.6) that

$$\begin{aligned}
&\mathbb{E}U(X^\xi(t_\alpha) - X^\eta(t_\alpha)) \\
&\le \mathbb{E}U(\xi - \eta) - \lambda_2 \mathbb{E}\int_0^{t_\alpha} U(X^\xi(s) - X^\eta(s))ds \\
&\le \mathbb{E}U(\xi - \eta) - c_2\lambda_2 \mathbb{E}\int_0^{t_\alpha} \|X^\xi(s) - X^\eta(s)\|_H^2 ds \\
&\le \mathbb{E}U(\xi - \eta) - c_2\lambda_2\alpha^2 \mathbb{E}(\tau_\alpha \wedge \tau_\beta \wedge t).
\end{aligned}$$

Sketch of Proof

Hence

$$P\{\tau_\alpha \wedge \tau_\beta \geq t\} \leq \frac{\mathbb{E}U(\xi - \eta)}{c_2 \lambda_2 \alpha^2 t},$$

which furthermore implies that for given $\epsilon \in (0, 1)$ there exists a constant $T = T(\mathcal{K}, \epsilon) > 0$ such that

$$P\{\tau_\alpha \wedge \tau_\beta \leq T\} > 1 - \frac{\epsilon}{4}, \qquad (2.12)$$

which yields

$$P\{\tau_\alpha \leq T\} \geq 1 - \frac{\epsilon}{2}. \qquad (2.13)$$

Sketch of Proof

Now, define stopping time

$$\sigma = \inf\{t \geq \tau_\alpha \wedge T : \|X^\xi(t) - X^\eta(t)\|_H \geq \epsilon\}.$$

Let $t > T$, we have

$$P\{\tau_\alpha \leq T, \sigma \leq t\} < \frac{\epsilon}{2}. \tag{2.14}$$

While, by (2.13) and (2.14)

$$P\{\sigma \leq t\} \leq P\{\tau_\alpha \leq T, \sigma \leq t\} + P\{\tau_\alpha > T\} < \epsilon.$$

Letting $t \to \infty$, we have

$$P\{\sigma < \infty\} \leq \epsilon.$$

This implies that for any $\xi, \eta \in \mathcal{K}$, we must have that for $t \geq T$

$$P\{\|X^\xi(t) - X^\eta(t)\|_H < \epsilon\} \geq 1 - \epsilon,$$

Let $\mathcal{P}(\mathcal{H})$ denote all probability measures on H. For $P_1, P_2 \in \mathcal{P}(\mathcal{H})$ define metric $d_{\mathbb{L}}$

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{f \in L} \left| \int_H f(x) P_1(dx) - \int_H f(x) P_2(dx) \right|$$

and

$$\mathbb{L} = \{ f : H \to R : |f(x) - f(y)| \leq \|x - y\|_H \;\; and \;\; |f(\cdot)| \leq 1 \}$$

Lemma

Let (2.8) hold. Then, for any compact subset $\mathcal{K}$ of H,

$$\lim_{t\to\infty} d_{\mathbb{L}}(P(t,\xi,\cdot),P(t,\zeta,\cdot)) = 0, \quad \text{uniformly in } \xi,\zeta\in\mathcal{K}. \tag{3.1}$$

Lemma

Let (2.3) and (2.8) hold. Then, $\{P(t,\xi,\cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}(\mathcal{H})$ for any $\xi \in H$.

Lemma

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Lemma

Let (2.3) and (2.8) hold. Then, $\{P(t,\xi,\cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}(\mathcal{H})$ for any $\xi \in H$.

Theorem

Under the conditions of Lemma 3.1 and Lemma 3.2, the mild solution $X(t)$ to (1.1) is stable in distribution.

Consider the following semi-linear stochastic partial differential equation

$$\begin{cases} dy(x,t) = \frac{\partial^2}{\partial x^2} y(x,t)dt + \sigma f(y(x,t))dW(t), & t \geq 0, 0 < x < 1, \\ y(0,t) = y(1,t) = 0, \quad t \geq 0; \quad y(x,0) = y_0(x), & 0 \leq x \leq 1, \end{cases} \tag{4.1}$$

where $W(t), t \geq 0$, is a real standard Brownian motion, σ is a real number and f is a real Lipschitz continuous function on $L^2(0,1)$ satisfying for $u, v \in L^2(0,1)$ and some positive constants c, k

$$\begin{aligned} |f(u)| &\leq c(\|u\|_H + 1), \\ |f(u) - f(v)| &\leq k\|u - v\|_H. \end{aligned} \tag{4.2}$$

we take $H = L^2(0,1)$ and $A = \frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H_0^1(0,1) \bigcap H^2(0,1)$. Then for any $u \in \mathcal{D}(A)$

$$\langle u, Au \rangle_H \leq -\pi^2 \|u\|_H^2.$$

For $\forall u \in \mathcal{D}(A)$

$$\mathcal{L}\|u\|_H^2 = 2\langle u, Au \rangle_H + \sigma^2 |f(u)|^2 \leq -2(\pi^2 - \sigma^2 c^2)\|u\|_H^2 + 2\sigma^2 c^2, \quad .$$

Similarly,

$$\mathcal{L}\|u - v\|_H^2 \leq -(2\pi^2 - \sigma^2 k^2)\|u - v\|_H^2.$$

Therefore, if $\sigma^2 c^2 < \pi^2$ and $\sigma^2 k^2 < 2\pi^2$, then we immediately deduce by Theorem 10 that the mild solution process $y(x,t)$ of (4.1) is stable in distribution.

$$dX(t) = [AX(t) + F(X(t), X(t-\tau))]dt + G(X(t), X(t-\tau))dW(t) + \int_{\mathbb{Z}} L(X(t), X(t-\tau), u)\tilde{N}(dt, du) \tag{5.1}$$

Lemma

Assume there exist constants $\lambda_1 > \lambda_2 \geq 0$ *and* $\beta \geq 0$ *such that for any* $x, y \in \mathcal{D}(A)$

$$2\langle x, Ax + F(x,y)\rangle_H + \|G(x,y)\|^2_{\mathcal{L}_2^0} + \int_{\mathbb{Z}} \|L(x,y,u)\|_H^2 \lambda(du) \leq -\lambda_1 \|x\|_H^2 + \lambda_2 \|y\|_H^2 + \beta. \tag{5.2}$$

Then

$$\sup_{0 \leq t < \infty} \mathbb{E}\|X_t^\xi\|_D^2 < \infty \quad \forall \xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; H). \tag{5.3}$$

Lemma

Assume that there are constants $\lambda_3 > \lambda_4 \geq 0$ *such that for any* $x, y, z_1, z_2 \in \mathcal{D}(A)$

$$2\langle x - y, A(x-y) + F(x, z_1) - F(y, z_2)\rangle_H + \|G(x, z_1) - G(y, z_2)\|^2_{\mathcal{L}_2^0}$$
$$+ \int_{\mathbb{Z}} \|L(x, z_1, u) - L(y, z_2, u)\|^2_H \lambda(du)$$
$$\leq -\lambda_3 \|x - y\|^2_H + \lambda_4 \|z_1 - z_2\|^2_H. \tag{5.4}$$

Then for any compact subset $\mathcal{K}$ *of* $D([-\tau, 0]; H)$

$$\lim_{t\to\infty} \mathbb{E}\|X_t^\xi - X_t^\eta\|^2_D = 0 \quad \textit{uniformly in } \xi, \eta \in \mathcal{K}. \tag{5.5}$$

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Thank You