Critical Brownian sheet does not have double points

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Outline

- The Brownian sheet and multiple points
- No double points in the critical case
- Hitting probability of M_2
- Hausdorff dimension of $M_2 \cap A$

1. The Brownian sheet and multiple points

The Brownian sheet $B = \{B(t),\, t \in \mathbb{R}^N_+\}$ is a centered (N,d)-Gaussian field whose covariance function is

$$\mathbb{E}\big[B_i(s)B_j(t)\big] = \delta_{ij} \prod_{k=1}^{N} s_k \wedge t_k,$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

- When $N = 1$, B is Brownian motion in $\mathbb{R}^d$.
- B is $N/2$-self-similar, but it does not have stationary increments.

Multiple points of the Brownian sheet

Recall that $x \in \mathbb{R}^d$ is a *k-multiple point of B* if there exist distinct points $t^1, \ldots, t^k \in (0, \infty)^N$ such that $B(t^1) = \cdots = B(t^k) = x$.

We write M_k for the set of all k-multiple points of B. Note that $M_{k+1} \subseteq M_k$ for all $k \geq 2$.

We may also consider the set of k-multiple times:

$$L_k = \{(t^1, \cdots, t^k) \in \mathbb{R}^{kN}_{\neq} : \, B(t^1) = \cdots = B(t^k)\}.$$

Some known results and questions

When $N = 1$, i.e., B is Brownian motion, there is a large literature on

- the existence of multiple points, starting with the works of Dvoretzky, Erdös and Kakutani (1950–).
- fractal properties of M_k and L_k: Taylor (1966), LeGall (1986, 1987), . . .
- self-intersection local times: Wolpert (1978), Geman, Horowitz and Rosen (1984), Dynkin (1985, 1986, 1987, 1988), ...
- See Taylor (1986) and X. (2004) for surveys.

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For $N > 1$, it is known [Rosen, 1984; Khoshnevisan, 1997] that

- If $Nk < (k-1)d/2$, then $M_k = \varnothing$ a.s.
- If $Nk > (k-1)d/2$, then $M_k \neq \varnothing$ a.s.

When $Nk > (k-1)d/2$, Rosen (1984) showed that

$$\dim_{\rm H} L_k = Nk - (k-1)d/2 \qquad \text{a.s.}$$

When $d = 1$, Zhou (1994) studied the exact Hausdorff measure of L_k.
The Hausdorff dimension of M_k was determined by Chen Xiong (1994), Khoshnevisan, Wu and X. (2006):

$$\dim_{\rm H} M_k = d - k(d-2N)^+ \qquad \text{a.s.}$$

where $a^+ = \max\{0, a\}$.

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The following conjecture and questions are interesting, most of them are open:

1. If $Nk = (k-1)d/2$, then $M_k = \varnothing$ a.s.
2. Determine hitting probability for M_k. That is, provide explicit conditions on $F \subset \mathbb{R}^d$ such that

$$\mathbb{P}\{M_k \cap F \neq \varnothing\} > 0.$$

3. Similar question for L_k.
4. Find exact Hausdorff and packing measure functions for M_k and L_k (if exist).

In this paper, we solve Problems (1) and (2) for $k = 2$.

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In this paper, we solve Problems (1) and (2) for $k = 2$.

2. No double points in the critical case

The following is our main result.

Theorem (2.1)

An N-parameter, d-dimensional Brownian sheet has double points if and only if $2(d-2N) < d$.

In addition, M_2 has positive Lebesgue measure almost surely if and only if $d < 2N$.

The first part shows that B has no double points when $2N = d/2$. The second verifies a conjecture of Fristedt (1995).

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A decoupling theorem

Denote $\mathcal{T} := \left\{(s\,,t) \in (0\,,\infty)^{2N} :\ s_i \neq t_i,\ i = 1,\ldots,N\right\}.$

Theorem (2.2)

Choose and fix a Borel set $A \subseteq \mathbb{R}^d$. Then,

$$\mathbb{P}\left\{\exists\,(t^1\,,t^2) \in \mathcal{T} :\ B(t^1) = B(t^2) \in A\right\} > 0 \qquad (1)$$

if and only if

$$\mathbb{P}\left\{\exists\,(t^1\,,t^2) \in \mathcal{T} :\ W_1(t^1) = W_2(t^2) \in A\right\} > 0, \qquad (2)$$

where W_1 and W_2 are independent N-parameter Brownian sheets in $\mathbb{R}^d$ (unrelated to B).

For proving Theorem 2.2, we develop a conditional potential theory for the Brownian sheet B.

We start with some notation.

- We identify subsets of $\{1, \ldots, N\}$ with partial orders on $\mathbb{R}^N$ as follows:
  For all $s, t \in \mathbb{R}^N$ and $\pi \subseteq \{1, \ldots, N\}$,

$$s \prec_\pi t \quad \text{iff} \quad \begin{cases} s_i \le t_i & \text{for all } i \in \pi, \\ s_i \ge t_i & \text{for all } i \notin \pi. \end{cases}$$

- Every s and t in $\mathbb{R}^N$ can be compared via some π.
- We write $s \prec_\pi t$ and $t \succ_\pi s$ interchangeably.

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- We write $s \curlywedge_\pi t$ for the N-vector whose jth coordinate is $\min(s_j, t_j)$ if $j \in \pi$ and $\max(s_j, t_j)$ otherwise.
- Given a partial order π and a point $s \in \mathbb{R}^N_+$, we define $\mathcal{F}_\pi(s)$ to be the σ-algebra generated by $\{B(u),\ u \prec_\pi s\}$ and all P-null sets. We then make the filtration $(\mathcal{F}_\pi(s),\ s \in \mathbb{R}^N_+)$ right-continuous in the partial order π, so that $\mathcal{F}_\pi(s) = \cap_{t \succ_\pi s} \mathcal{F}_\pi(t)$.
- Let $\mathbb{P}^\pi_s$ be a regular conditional distribution for B given $\mathcal{F}_\pi(s)$ and let

$$\mathbb{E}^\pi_s f := \int f \, d\mathbb{P}^\pi_s = \mathbb{E}(f \mid \mathcal{F}_\pi(s))$$

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Theorem (2.3)

Choose and fix an upright box $\Theta := \prod_{j=1}^{N}[a_j, b_j]$ *in* $(0,\infty)^N$. *For any partial order* $\pi \subseteq \{1,\ldots,N\}$, *choose and fix some vector* $s \in (0,\infty)^N \setminus \Theta$ *such that* $s \prec_\pi t$ *for every* $t \in \Theta$. *Then for all* $\mathcal{F}_\pi(s)$*-measurable bounded random sets* A,

$$\mathbb{P}_s^\pi\left\{B(u) \in A \text{ for some } u \in \Theta\right\} \asymp \mathrm{Cap}_{d-2N}(A), \qquad (3)$$

where $Z_1 \asymp Z_2$ *means* $\mathbb{P}\left\{\mathbf{1}_{\{Z_1>0\}} = \mathbf{1}_{\{Z_2>0\}}\right\} = 1$.

This theorem generalizes Theorem 1.1 of Khoshnevisan and Shi (1999).

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Analysis of pinned sheets

For all $s \in (0\,,\infty)^N$ and $t \in \mathbb{R}_+^N$, define

$$B_s(t) := B(t) - \delta_s(t)B(s), \tag{4}$$

where

$$\delta_s(t) := \prod_{j=1}^{N} \left(\frac{s_j \wedge t_j}{s_j} \right). \tag{5}$$

It is not too difficult to see that

$$B_s(t) = B(t) - \mathbb{E}\left[B(t)\,\middle|B(s)\right]. \tag{6}$$

Proof of Theorem 2.3 replies on the following lemma.

Lemma (Khoshnevisan and X., 2007)

Choose and fix a partial order $\pi \subseteq \{1, \ldots, N\}$ *and a time point* $s \in (0, \infty)^N$. *Then* $\{B_s(t)\}_{t \succ_\pi s}$ *is independent of* $\mathcal{F}_\pi(s)$.

Moreover, for every nonrandom upright box $I \subset (0, \infty)^N$ *and* $\pi \subseteq \{1, \ldots, N\}$, *there exists a constant* $c > 1$ *such that for all* $s, u, v \in I$,

$$c^{-1}\|u - v\| \le \mathrm{Var}\left(B_s^1(u) - B_s^1(v)\right) \le c\|u - v\|, \qquad (7)$$

where $B_s^1(t)$ *denotes the first coordinate of* $B_s(t)$ *for all* $t \in \mathbb{R}_+^N$.

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Proof of Theorem 2.2

Observe that there exist distinct points s and $t \in \mathcal{T}$ such that $B(s) = B(t) \in A$, if and only if we can find disjoint closed upright boxes Θ_1 and Θ_2, with vertices with rational coordinates, such that $B(\Theta_1) \cap B(\Theta_2) \cap A \neq \varnothing$.

There exist $s \in (0, \infty)^N$ and a partial order $\pi \subseteq \{1, \ldots, N\}$ such that

- $u \prec_\pi s$ for all $u \in \Theta_1$ and
- $s \prec_\pi v$ for all $v \in \Theta_2$.

Applying Theorem 2.3, we derive

$$\begin{aligned} &\mathbb{P}\left\{B(\Theta_1) \cap B(\Theta_2) \cap A \neq \varnothing\right\} > 0 \\ \Leftrightarrow\ &\mathbb{E}\left[\mathrm{Cap}_{d-2N}\left(B(\Theta_1) \cap A\right)\right] > 0. \end{aligned}$$

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Proof of Theorem 2.2

Observe that there exist distinct points s and $t \in \mathcal{T}$ such that $B(s) = B(t) \in A$, if and only if we can find disjoint closed upright boxes Θ_1 and Θ_2, with vertices with rational coordinates, such that $B(\Theta_1) \cap B(\Theta_2) \cap A \neq \varnothing$.

There exist $s \in (0, \infty)^N$ and a partial order $\pi \subseteq \{1, \ldots, N\}$ such that

- $u \prec_\pi s$ for all $u \in \Theta_1$ and
- $s \prec_\pi v$ for all $v \in \Theta_2$.

Applying Theorem 2.3, we derive

$$\mathbb{P}\left\{B(\Theta_1) \cap B(\Theta_2) \cap A \neq \varnothing\right\} > 0$$

$$\Leftrightarrow \ \mathbb{E}\left[\mathrm{Cap}_{d-2N}\left(B(\Theta_1) \cap A\right)\right] > 0.$$

The main result of Khoshnevisan and Shi (1999) is that

$$\mathrm{Cap}_{d-2N}(E) > 0 \;\Leftrightarrow\; \mathbb{P}\{W_2(\Theta_2) \cap E \neq \varnothing\} > 0,$$

where W_2 is a Brownian sheet that is independent of B.

We apply this with $E := B(\Theta_1) \cap A$ first, and then $E := W_2(\Theta_2) \cap A$ to deduce that

$$\begin{aligned} &\mathbb{P}\left\{B(\Theta_1) \cap B(\Theta_2) \cap A \neq \varnothing\right\} > 0 \\ &\quad\Leftrightarrow\; \mathbb{P}\left\{B(\Theta_1) \cap W_2(\Theta_2) \cap A \neq \varnothing\right\} > 0 \\ &\quad\Leftrightarrow\; \mathbb{E}\left[\mathrm{Cap}_{d-2N}\left(W_2(\Theta_2) \cap A\right)\right\} > 0 \\ &\quad\Leftrightarrow\; \mathbb{P}\left\{W_1(\Theta_1) \cap W_2(\Theta_2) \cap A \neq \varnothing\right\} > 0, \end{aligned}$$

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Finally, since the family of pairs of such closed upright boxes Θ_1 and Θ_2 is countable, it proves Theorem 2.2.

Remarks

- The proof depends on ordering disjoint upright boxes Θ_1 and Θ_2.
- This can be done for disjoint upright boxes $\Theta_1, \Theta_2, \Theta_3$ in $\mathbb{R}^2$. So Theorem 2.2 and Theorem 2.1 hold for $N = 2$ and $k = 3$.
- However, this can not be done in general for disjoint upright boxes $\Theta_1, \ldots, \Theta_k$ in $\mathbb{R}^N$ with $N \geq 3$ and $k \geq 3$.

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3. Hitting probabilities of M_2

Theorem (3.1)

Let A denote a nonrandom Borel set in $\mathbb{R}^d$.

- *If $d > 2N$, then $\mathbb{P}\{M_2 \cap A \neq \varnothing\} > 0$ if and only if $\mathrm{Cap}_{2(d-2N)}(A) > 0$, where Cap_β denotes the Bessel–Riesz capacity in dimension $\beta \in \mathbb{R}$.*
- *If $d = 2N$, then $\mathbb{P}\{M_2 \cap A \neq \varnothing\} > 0$ if and only if $\exists$ $\mu \in \mathcal{P}(A)$, such that*

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- *Finally, if* $d < 2N$*, then* $\mathbb{P}\{M_k \cap A \neq \varnothing\} > 0$ *for all* $k \geq 2$ *and all nonvoid nonrandom Borel sets* $A \subset \mathbb{R}^d$.

Theorem 3.1 follows from Theorem 2.2, the main theorem in Khoshnevisan and Shi (1999) and a result of Peres (1997).

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4. Hausdorff dimension of $M_2 \cap A$

Now for any closed set $A \subset \mathbb{R}^d$, we consider $\dim_{\mathrm{H}}(M_2 \cap A)$.

Theorem (4.1)

Choose and fix a nonrandom closed set $A \subset \mathbb{R}^d$.

- *If* $\dim_{\mathrm{H}} A < 2(d - 2N)$, *then* $\mathbb{P}\big(M_2 \cap A = \varnothing\big) = 1$.
- *If* $\dim_{\mathrm{H}} A \geq 2(d - 2N)$, *then*

$$\|\dim_{\mathrm{H}}(M_2 \cap A)\|_{L^\infty(\mathbb{P})} = \dim_{\mathrm{H}} A - 2(d - 2N)^+,$$

where

$$\|Z\|_{L^\infty(\mathbb{P})} := \inf\{\lambda > 0 : \mathbb{P}\{Z > \lambda\} = 0\}, \ (\inf \varnothing := 0)$$

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The proof has two parts:

- The upper bound: it can be proved that

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Thank you