Weakly Coupled Lévy Type Operators and Switched Lévy Type Processes

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Joint work with Zhen-Qing Chen

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Background

- Switched Diffusion Processes are often used to describe some typical hybrid systems: systems with multiple modes.
  - Stability;
  - Ergodicity;
  - Harnack Inequality;
  - $\cdots\cdots$.
- Lévy Type Processes (or Jump-Diffusion Processes) can be considered as continuous Itô diffusions perturbed by random jumps.
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In this work, we will consider switched Lévy type processes.

Roughly, a switched Lévy type process is a combination of switched diffusion process and Lévy type process.

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Weakly Coupled Lévy Type Operator

Let d and N be two positive integers and set $\mathbb{S} := \{1, 2, \cdots, N\}$. For certain "nice" functions

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} : \mathbb{R}^d \mapsto \mathbb{R}^N,$$

consider the following weakly coupled Lévy type operator:

$$Sf = \begin{pmatrix} \mathcal{L}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{L}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{L}_N \end{pmatrix} f + Qf. \tag{1}$$

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Here, for each $k \in \mathbb{S}$, $\mathcal{L}_k$ is a Lévy type operator defined as follows:

$$\mathcal{L}_k f_k(x) = \frac{1}{2}\mathrm{tr}\big(a(x,k)\mathrm{Hess} f_k(x)\big) + \langle b(x,k), \nabla f_k(x)\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \big(f_k(x+u) - f_k(x) - \langle \nabla f_k(x), u\rangle \mathbf{1}_{\{|u|\le\varepsilon_0\}}\big)\nu(x,k,du), \tag{2}$$

where for each $(x,k) \in \mathbb{R}^d \times \mathbb{S}$,

$a(x,k) = \big(a_{ij}(x,k)\big)$ is a nonnegative definite symmetric $d \times d$-matrix,

$b(x,k) = \big(b_i(x,k)\big)$ is a d-dimensional vector,

and $\nu(x,k,\cdot)$ is a Lévy kernel satisfying that $\nu(x,k,\cdot)$ is a nonnegative σ-finite measure on $\mathbb{R}^d \setminus \{0\}$ such that there exists a nonnegative constant ε_0 satisfying that $\int_{\mathbb{R}^d\setminus\{0\}} |u|^2 \mathbf{1}_{\{|u|\le\varepsilon_0\}}\nu(x,k,du) < \infty$ and $\nu(x,k,\mathbb{R}^d \setminus \{|u| \le \varepsilon_0\}) < \infty$.

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And here, $Q = (q_{kl}(x))$ is an $N \times N$ matrix-valued measurable function on $\mathbb{R}^d$ such that

$$q_{kl}(x) \geq 0 \quad \text{on} \quad \mathbb{R}^d \quad \text{for} \quad k \neq l \tag{3}$$

and, for each fixed $k \in \mathbb{S}$,

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Switched Lévy Type Process

As is well known, under some mild conditions, a Lévy type operator as in (2) can be associated with a regular (i.e. non-explosive) Lévy type process.

For the existence and uniqueness of the strong Markov process $Y = (X, \Lambda)$ corresponding to the weakly coupled Lévy type operator S defined in (1), we make the following assumption.

Assumption 1

Assume that for each $k \in \mathbb{S}$, the Lévy type operator $\mathcal{L}_k$ defined in (2) is associated with a regular (i.e. non-explosive) Lévy type process $\widetilde{X}^{(k)}$. Moreover, assume that for each $k \in \mathbb{S}$, the function $q_{kk}(x) \leq 0$ is bounded from below.

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The First Switching Time

We will use the Ikeda-Nagasawa-Watanabe piecing together procedure to construct the strong Markov process $Y = (X, \Lambda)$ which has infinitesimal generator S defined by (1).

Let

$$\tau = \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\} \qquad (5)$$

be the first switching time of Y.

Clearly, we can identify $Y(t) = (X(t), \Lambda(t))$ with $(\widetilde{X}^{(k)}(t), \Lambda(0))$ for $0 \leq t < \tau$ when $(X(0), \Lambda(0)) = (x, k)$, where $\widetilde{X}^{(k)}$ just is the Lévy type process introduced in Assumption 1.

Meanwhile, we also have that $\mathbb{P}^{(x,k)}(\tau > 0) = 1$ for all $(x, k) \in \mathbb{R}^d \times \mathbb{S}$.

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Killed Lévy Type Processes

To construct $Y = (X, \Lambda)$, we need to introduce a family of killed Lévy type processes.

For each $k \in \mathbb{S}$, we kill the Lévy type process $\widetilde{X}^{(k)}$ at the rate $(-q_{kk})$,

$$\begin{aligned} \mathbb{E}_k^{(x)}[f(X^{(k)}(t))] &= \mathbb{E}_k^{(x)}[f(\widetilde{X}^{(k)}(t)); t < \tau] \\ &= \mathbb{E}_k^{(x)}[\exp\{\int_0^t q_{kk}(\widetilde{X}^{(k)}(s))ds\} f(\widetilde{X}^{(k)}(t))], \end{aligned} \quad (6)$$

to get a subprocess $X^{(k)}$.

Equivalently, $X^{(k)}$ can be defined as $X^{(k)}(t) = \widetilde{X}^{(k)}(t)$ if $t < \tau$ and $X^{(k)}(t) = \partial$ if $t \geq \tau$, where ∂ is a cemetery point added to $\mathbb{R}^d$.

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By the Ikeda-Nagasawa-Watanabe piecing together procedure, now we construct the switched Lévy type process $Y = (X, \Lambda)$ on $\mathbb{R}^d \times \mathbb{S}$ as follows.

For given $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, let $Y(t) = (X^{(k)}(t), \Lambda(0))$ for $0 \le t < \tau^{(1)}$, where $X^{(k)}(0) = x$, $\Lambda(0) = k$ and $\tau^{(1)} = \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\}$.

Next, we select an $l \in \mathbb{S} \setminus \{k\}$ with the probability distribution $(-q_{kl}/q_{kk})(X(\tau^{(1)}-))$ and glue a copy of $Y = (X, \Lambda)$ starting from $(X(\tau^{(1)}-), l)$.

Namely, define $Y(\tau^{(1)} + t) = (X^{(l)}(t), \Lambda(0))$ for $0 \le t < \tau^{(2)}$, where $X^{(l)}(0) = X(\tau^{(1)}-)$, $\Lambda(0) = l$ and $\tau^{(2)} = \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\}$.

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By the Ikeda-Nagasawa-Watanabe piecing together procedure, now we construct the switched Lévy type process $Y = (X, \Lambda)$ on $\mathbb{R}^d \times \mathbb{S}$ as follows.

For given $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, let $Y(t) = (X^{(k)}(t), \Lambda(0))$ for $0 \le t < \tau^{(1)}$, where $X^{(k)}(0) = x$, $\Lambda(0) = k$ and $\tau^{(1)} = \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\}$.

Next, we select an $l \in \mathbb{S} \setminus \{k\}$ with the probability distribution $(-q_{kl}/q_{kk})(X(\tau^{(1)}-))$ and glue a copy of $Y = (X, \Lambda)$ starting from $(X(\tau^{(1)}-), l)$.

Namely, define $Y(\tau^{(1)} + t) = (X^{(l)}(t), \Lambda(0))$ for $0 \le t < \tau^{(2)}$, where $X^{(l)}(0) = X(\tau^{(1)}-)$, $\Lambda(0) = l$ and $\tau^{(2)} = \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\}$.

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Y Has Infinitesimal Generator S

Theorem 2

Suppose that Assumption 1 holds. Then, the process Y constructed above is a switched Lévy type process on $\mathbb{R}^d \times \mathbb{S}$ associated the infinitesimal generator S in (1).

Sketched Proof. Denote by $\{P_t, t \geq 0\}$ and $\{G_\alpha, \alpha > 0\}$, respectively, the transition semigroup and resolvent of the switched process $Y = (X, \Lambda)$.

For each $k \in \mathbb{S}$, let $\{G_\alpha^{(k)}, \alpha > 0\}$ be the resolvent for the generator $\mathcal{L}_k + q_{kk}$.

Let $f(x, k) \geq 0$ on $\mathbb{R}^d \times \mathbb{S}$.

Applying the strong Markov property at the first switching time τ and recalling the construction of Y, we obtain

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Applying the strong Markov property at the first switching time τ and recalling the construction of Y, we obtain

Y Has Infinitesimal Generator S (Cont)

$$
\begin{aligned}
G_\alpha f(x,k) &= \mathbb{E}^{(x,k)}\left[\int_0^\infty e^{-\alpha t} f(X_t,\Lambda_t)dt\right] \\
&= \mathbb{E}^{(x,k)}\left[\int_0^\tau e^{-\alpha t} f(X_t,k)dt\right] + \mathbb{E}^{(x,k)}\left[\int_\tau^\infty e^{-\alpha t} f(X_t,\Lambda_t)dt\right] \\
&= G_\alpha^{(k)} f(x,k) + \mathbb{E}^{(x,k)}\left[e^{-\alpha\tau} G_\alpha f(X_\tau,\Lambda_\tau)\right] \\
&= G_\alpha^{(k)} f(x,k) + \sum_{l\in\mathbb{S}\setminus\{k\}} \mathbb{E}^{(x,k)}\left[e^{-\alpha\tau}\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-}) G_\alpha f(X_{\tau-},l)\right] \\
&= G_\alpha^{(k)} f(x,k) + \sum_{l\in\mathbb{S}\setminus\{k\}} G_\alpha^{(k)}(q_{kl} G_\alpha f(\cdot,l))(x).
\end{aligned}
$$

Hence for each fixed $k \in \mathbb{S}$,

$$
\mathcal{L}_k G_\alpha f(\cdot,k) + \sum_{l\in\mathbb{S}} q_{kl} G_\alpha f(\cdot,l) - \alpha G_\alpha f(\cdot,k) = -f(\cdot,k).
$$

This shows that S defined in (1) is the infinitesimal generator of Y.

Dynkin's Formula

Remark 3

For each $k \in \mathbb{S}$, and for any function $f(\cdot, k) \in C_0^2(\mathbb{R}^d)$, define an operator $\mathcal{A}$ as follows:

$$\mathcal{A}f(x,k) = \mathcal{L}_k f(x,k) + Q(x)f(x,k), \tag{7}$$

where operator $Q(x)$ is further defined as

$$Q(x)f(x,k) = \sum_{l \in \mathbb{S}} q_{kl}(x)\big(f(x,l) - f(x,k)\big). \tag{8}$$

Since S is the infinitesimal generator of Y by Theorem 2, we know that for each bounded stopping time τ for the process Y and each function $f(\cdot, k) \in C_0^2(\mathbb{R}^d)$ with $k \in \mathbb{S}$, the Dynkin's formula

$$\mathbb{E}^{(x,k)} f(X(\tau), \Lambda(\tau)) = f(x,k) + \mathbb{E}^{(x,k)} \int_0^\tau \mathcal{A}f(X(s), \Lambda(s))ds \tag{9}$$

Successive Switching Instants

We now consider the non-explosiveness of the process $Y = (X, \Lambda)$.

To do so, we need to prove that $Y = (X, \Lambda)$ can only have finitely many switches during a finite time interval.

Let us denote the successive switching instants of (X, Λ) by

$$\begin{aligned} \tau_1 &= \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\}, \\ \tau_n &= \inf\{t : t > \tau_{n-1}, \Lambda(t) \neq \Lambda(\tau_{n-1})\}, \; n \geq 2, \end{aligned} \tag{10}$$

with the convention that $\inf \emptyset = +\infty$.

Moreover, we also define a counting process, $v(t) := \max\{n : \tau_n \leq t\}$.

Obviously, τ_1 just is the first switching time τ defined in (5) and $v(t)$ is the total number of switches of (X, Λ) prior to t.

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Exit Times

Lemma 4

Suppose that Assumption 1 holds. For every $(x,k) \in \mathbb{R}^d \times \mathbb{S}$ *and every* $t > 0$, *we have that*

$$\mathbb{P}^{(x,k)}(\lim_{n\to\infty} \tau_n = \infty) = 1, \qquad \mathbb{P}^{(x,k)}\big(v(t) < +\infty\big) = 1. \tag{11}$$

We now proceed to prove the non-explosiveness of $Y = (X, \Lambda)$. For all integers $m \geq 1$, let $O_m := \{x \in \mathbb{R}^d : |x| < m\}$ and define

$$\gamma_m := \inf\{t \geq 0 : (X(t), \Lambda(t)) \in O_m^c \times \mathbb{S}\}.$$

It follows from the second equality in (11) that for every $(x,k) \in \mathbb{R}^d \times \mathbb{S}$ and every $t > 0$, $\mathbb{P}^{(x,k)}\big(v(t) < +\infty\big) = 1$. Consequently, for every $m \geq 1$, we then have that

$$\mathbb{P}^{(x,k)}\big(\gamma_m \leq t\big) = \sum_{n=0}^{\infty} \mathbb{P}^{(x,k)}\big(\gamma_m \leq t,\, v(t) = n\big). \tag{12}$$

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Non-Explosiveness

Theorem 5

Suppose that Assumption 1 holds. The process $Y = (X, \Lambda)$ is non-explosive.

Remark 6

We can define truncations $(X(t \wedge \gamma_m), \Lambda(t \wedge \gamma_m))$ $(m \geq 1)$ of the non-explosive process (X, Λ). Then we have that for every $m \geq 1$,

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holds for each point $(x, k) \in O_m \times \mathbb{S}$ and each function $f(\cdot, k) \in C^2(\mathbb{R}^d)$ with $k \in \mathbb{S}$.

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Some Conditions

We will prove the Feller property for the switched Lévy type process $Y = (X, \Lambda)$ under some conditions.

Assumption 7

Assume that for $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, the Lévy kernel $\nu(x, k, \cdot)$ is a nonnegative finite measure on $\mathbb{R}^d \setminus \{0\}$. Moreover, assume that there exists a constant $H > 0$ such that

$$|\sigma(x, k) - \sigma(y, k)| + |b(x, k) - b(y, k)| \leq H|x - y| \tag{14}$$

and

$$\int_{\mathbb{R}^d \setminus \{0\}} |z| \|\nu(x, k, \cdot) - \nu(y, k, \cdot)\|(dz) \leq H|x - y| \tag{15}$$

for all $x, y \in \mathbb{R}^d$ and $k \in \mathbb{S}$, where $\sigma(x, k)$ is a square root of $a(x, k)$ such that $a(x, k) = \sigma(x, k)\sigma(x, k)^*$.

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Feller Property

Using the coupling methods, we prove the following two lemmas.

Lemma 8

Suppose that Assumptions 1 and 7 hold. For each $k \in \mathbb{S}$, the Lévy type process $\widetilde{X}^{(k)}$ generated by the Lévy type operator $\mathcal{L}_k$ defined in (2) has Feller property.

Lemma 9

Suppose that Assumptions 1 and 7 hold. For each $k \in \mathbb{S}$, the killed Lévy type process $X^{(k)}$ introduced in (6) has Feller property.

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By virtue of the relation of $Y = (X, \Lambda)$ and the killed Lévy type processes $X^{(k)}$, $k \in \mathbb{S}$, we prove the following theorem.

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Suppose that Assumptions 1 and 7 hold. Then, The process $Y = (X, \Lambda)$ has Feller property.

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Some Auxiliary Processes and Further Conditions

We now introduce some auxiliary processes as follows.

For each $k \in \mathbb{S}$, let $\widetilde{X}^{(k),0}$ be the unique strong Markov process generated by operator $\mathcal{L}_k^0$, which is defined for "nice" function g on $\mathbb{R}^d$ as follows:

$$\mathcal{L}_k^0 g(x) = \frac{1}{2}\mathrm{tr}\big(a(x,k)\mathrm{Hess}g(x)\big) + \langle b(x,k), \nabla g(x)\rangle. \tag{16}$$

Assumption 12

Assume that for each $k \in \mathbb{S}$, $\widetilde{X}^{(k),0}$ has a positive transition probability density $\widetilde{p}^{(k),0}(t,x,y)$ with respect to the Lebesgue measure. Moreover, assume that $q_{kl}(x) > 0$ for all $x \in \mathbb{R}^d$ and $k \neq l \in \mathbb{S}$.

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Irreducibility and Aperiodicity

Let us fix a probability measure $\mu(\cdot)$ that is equivalent to the product measure on $\mathbb{R}^d \times \mathbb{S}$ of the Lebesgue measure on $\mathbb{R}^d$ and the counting measure on $\mathbb{S}$.

Proposition 13

Suppose that Assumptions 1, 7 and 12 hold. Then (X, Λ) is μ-irreducible. Moreover, for any given $\delta > 0$, all compact subsets of $\mathbb{R}^d \times \mathbb{S}$ are petite for the δ-skeleton chain of (X, Λ).

Proposition 14

Suppose that Assumptions 1, 7 and 12 hold. Then (X, Λ) is a μ-irreducible, aperiodic T-process.

Irreducibility and Aperiodicity

Let us fix a probability measure $\mu(\cdot)$ that is equivalent to the product measure on $\mathbb{R}^d \times \mathbb{S}$ of the Lebesgue measure on $\mathbb{R}^d$ and the counting measure on $\mathbb{S}$.

Proposition 13

Suppose that Assumptions 1, 7 and 12 hold. Then (X, Λ) is μ-irreducible. Moreover, for any given $\delta > 0$, all compact subsets of $\mathbb{R}^d \times \mathbb{S}$ are petite for the δ-skeleton chain of (X, Λ).

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Exponential Ergodicity

We proceed to investigate the exponential ergodicity for strong Markov process $Y = (X, \Lambda)$.

For any positive function $\Psi(x,k) \geq 1$ defined on $\mathbb{R}^d \times \mathbb{S}$ and any signed measure $\nu(\cdot)$ defined on $\mathcal{B}(\mathbb{R}^d \times \mathbb{S})$ we write

$$\|\nu\|_\Psi = \sup\{|\nu(\Phi)| : |\Phi| \leq \Psi\}.$$

For a function $\infty > \Psi \geq 1$ on $\mathbb{R}^d \times \mathbb{S}$, Markov process (X, Λ) is said to be Ψ-exponentially ergodic if there exist a probability measure $\pi(\cdot)$, a constant $\theta < 1$ and a finite-valued function $\Theta(x,k)$ such that

$$\|P(t,(x,k),\cdot) - \pi(\cdot)\|_\Psi \leq \Theta(x,k)\theta^t$$

for all $t \geq 0$ and all $(x,k) \in \mathbb{R}^d \times \mathbb{S}$.

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Exponential Ergodicity (Cont)

Moreover, a nonnegative function $V(x,k)$ defined on $\mathbb{R}^d \times \mathbb{S}$ is called a norm-like function if $V(x,k) \to \infty$ as $|x| \to \infty$ for all $k \in \mathbb{S}$. A Foster-Lyapunov drift condition: for some α, $\beta > 0$ and a norm-like function $V(x,k)$ which is twice continuously differentiable in x,

$$\mathcal{A}V(x,k) \leq -\alpha V(x,k) + \beta \tag{17}$$

for $(x,k) \in \mathbb{R}^d \times \mathbb{S}$.

Theorem 15

Suppose that (17) and Assumptions 1, 7 and 12 hold. Then (X,Λ) is Ψ-exponentially ergodic with $\Psi(x,k) = V(x,k) + 1$ and $\Theta(x,k) = B(V(x,k)+1)$, where B is a finite constant. Meanwhile, (X,Λ) is Ψ-uniformly ergodic with $\Psi(x,k) = V(x,k)$.

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Thank You Very Much!