Burgers-KPZ type PDEs characterising the path-independent property of the density of the Girsanov transformation for SDEs

Jiang-Lun Wu

Department of Mathematics, Swansea University, UK

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Outline

[1] J.-L. Wu, W. Yang, On stochastic differential equations and generalised Burgers equation. Submitted.

[2] A. Truman, F.-Y. Wang, J.-L. Wu, W. Yang, A Burgers-KPZ type parabolic equation for the path-independence of the density of the Girsanov transformation. Submitted.

Given $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,\infty)})$. Consider the following SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

where
$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$,
$\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, and
B_t is d-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion.
It is well known that under the usual conditions of linear growth and locally Lipschitz for the coefficients b and σ, there exists a unique solution to the equation with given initial data X_0.

The celebrated Girsanov theorem provides a very powerful tool to solve SDEs under the name of the *Girsanov transformation or the transformation of the drift*. Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the following condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right)\right] < \infty, \quad \forall t > 0.$$

Then, by Girsanov theorem,

$$\exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right), \quad t \in [0, \infty)$$

is an $\{\mathcal{F}_t\}$-martingale. Furthermore, for $t \geq 0$, we define

$$Q_t := \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right) \cdot P$$

or equivalently in terms of the Radon-Nikodym derivative

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right).$$

Then, for any $T > 0$,

$$\tilde{B}_t := B_t - \int_0^t \gamma(s, X_s)ds, \quad 0 \leq t \leq T$$

is an $\{\mathcal{F}_t\}$-Brownian motion under the probability Q_T.
Moreover, X_t satisfies

$$dX_t = [b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)]dt + \sigma(t, X_t)d\tilde{B}_t, \quad t \geq 0.$$

Motivation from economics and finance

Now look at the Radon-Nikodym derivative

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right)$$

we see that generally $\frac{dQ_t}{dP}$ depends on the “history” of the path up to t (i.e., $\{X_s : 0 \le s \le t\}$)!

While in economics and finance studies, in particular to the optimal problem for the utility functions in an equilibrium market, it is a necessary requirement that $\frac{dQ_t}{dP}$ depends only on the state X_t, not on the whole “history” $\{X_s : 0 \le s \le t\}$.

So mathematically, one requires that the Radon-Nikodym derivative is in the form of

$$Z(X_t, t) = \frac{dQ_t}{dP}, \quad t \in [0, \infty).$$

We call this *the path-independent property* of the density of the Girsanov transformation. We'd like to present a characterisation of this property.

Assumptions:
(i) (*Non-degeneracy*) The coefficient σ satisfies that the matrix $\sigma(t, x)$ is invertible, for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$;

(ii) Specify the function γ by

$$\gamma(t, x) = -(\sigma(t, x))^{-1} b(t, x)$$

so that $b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t) = 0$, and hence we require b and σ satisfy

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |(\sigma(s, X_s))^{-1} b(s, X_s)|^2 ds\right)\right] < \infty, \quad \forall t > 0.$$

Thus the associated probability measure Q_t is determined by

$$\begin{aligned}\frac{dQ_t}{dP} &= \exp\left(-\int_0^t \langle(\sigma(s,X_s))^{-1}b(s,X_s), dB_s\rangle\right.\\ &\qquad \left.-\frac{1}{2}\int_0^t \left|(\sigma(s,X_s))^{-1}b(s,X_s)\right|^2 ds\right).\end{aligned}$$

Now set

$$\hat{Z}_t := -\ln\frac{dQ_t}{dP}$$

that is

$$\hat{Z}_t = \int_0^t \langle(\sigma(s,X_s))^{-1}b(s,X_s), dB_s\rangle + \frac{1}{2}\int_0^t \left|(\sigma(s,X_s))^{-1}b(s,X_s)\right|^2 ds.$$

Clearly, $\hat{Z}_t$ is a one dimensional stochastic process with the stochastic differential form

$$d\hat{Z}_t = \frac{1}{2}\left|(\sigma(t,X_t))^{-1}b(t,X_t)\right|^2 dt + \langle(\sigma(t,X_t))^{-1}b(t,X_t), dB_t\rangle .$$

Theorem 1

Let $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$
\begin{aligned}
v(t, X_t) &= v(0, X_0) + \frac{1}{2} \int_0^t \left| (\sigma(s, X_s))^{-1} b(s, X_s) \right|^2 ds \\
&\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle
\end{aligned}
$$

equivalently,

$$
\frac{dQ_t}{dP} = \exp\{v(0, X_0) - v(t, X_t)\}, \quad t \in [0, \infty)
$$

holds if and only if

Theorem 1 (cont'd)

$$b(t,x) = (\sigma\sigma^* \nabla v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

and v satisfies the following time-reversed Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\left\{\left[Tr(\sigma\sigma^* \nabla^2 v)\right](t,x) + |\sigma^* \nabla v|^2(t,x)\right\}$$

where $\nabla^2 v$ stands for the Hessian matrix of v with respect to the second variable.

Proof

Necessity Assume that there exists a scalar function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ which is C^1 with respect to the first variable and C^2 with respect to the second variable such that

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \frac{1}{2} \int_0^t \left| (\sigma(s, X_s))^{-1} b(s, X_s) \right|^2 ds \\ &\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned}$$

holds, then we have

$$dv(t, X_t) = \frac{1}{2} \left| (\sigma(t, X_t))^{-1} b(t, X_t) \right|^2 dt + \langle (\sigma(t, X_t))^{-1} b(t, X_t), dB_t \rangle .$$

Now viewing $v(t, X_t)$ as the composition of the deterministic $C^{1,2}$-function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ with the continuous semi-martingale X_t, we can apply Itô's formula to $v(t, X_t)$ and further with the help of our original SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

we have the following derivation

$$\begin{aligned} dv(t, X_t) &= \left\{ \frac{\partial}{\partial t} v(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, X_t) \right. \\ &\quad \left. + \langle b, \nabla v \rangle (t, X_t) \right\} dt + \langle (\sigma^* \nabla v)(t, X_t), dB_t \rangle \end{aligned}$$

since

$$\langle \nabla v(t, X_t), \sigma(t, X_t)dB_t \rangle = \langle \sigma^*(t, X_t)\nabla v(t, X_t), dB_t \rangle .$$

Now comparing this with the previously obtained

$$dv(t, X_t) = \frac{1}{2}\left|(\sigma(t, X_t))^{-1} b(t, X_t)\right|^2 dt + \langle(\sigma(t, X_t))^{-1} b(t, X_t), dB_t\rangle$$

and using the uniqueness of Doob-Meyer's decomposition of continuous semi-martingale, we conclude that the coefficients of dt and dB_t must coincide, respectively, namely

$$(\sigma^{-1} b)(t, X_t) = (\sigma^* \nabla v)(t, X_t)$$

$$\frac{1}{2}|(\sigma^{-1} b)(t, X_t)|^2 = \frac{\partial}{\partial t} v(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^* \nabla^2 v)](t, X_t) + \langle b, \nabla v\rangle(t, X_t)$$

holds for all $t > 0$.

Since our SDE is non-degenerate, the support of $X_t, t \in [0, \infty)$ is the whole space $\mathbb{R}^d$. Hence, the following two equalities

$$(\sigma^{-1} b)(t, x) = (\sigma^* \nabla) v(t, x)$$

$$\frac{1}{2} |(\sigma^{-1} b)(t, x)|^2 = \frac{\partial}{\partial t} v(t, x) + \langle b, \nabla v \rangle (t, x) + \frac{1}{2} [Tr(\sigma \sigma^* \nabla v)](t, x)$$

hold on $[0, \infty) \times \mathbb{R}^d$. From these equalities we derive

$$b(t, x) = (\sigma \sigma^* \nabla v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d$$

and v satisfies the Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} \left\{ \left[Tr(\sigma \sigma^* \nabla^2 v) \right] (t, x) + |\sigma^* \nabla v|^2 (t, x) \right\} .$$

Sufficiency Assume that there exists a $C^{1,2}$ scalar function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ solving the Burgers-KPZ equation. Specify the drift b of the original SDE via

$$b(t, x) = (\sigma\sigma^* \nabla v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d .$$

We then have

$$\begin{aligned} dv(t, X_t) &= \Big[-\frac{1}{2}|\sigma^* \nabla v|^2(t, X_t) + \langle b, \nabla v\rangle(t, X_t)\Big] dt \\ &\quad + \langle(\sigma^* \nabla v)(t, X_t), dB_t\rangle \\ &= \frac{1}{2}|\sigma^{-1} b|^2(t, X_t)dt + \langle(\sigma^{-1} b)(t, X_t), dB_t\rangle . \end{aligned}$$

This clearly implies

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \frac{1}{2}\int_0^t \left|(\sigma(s, X_s))^{-1} b(s, X_s)\right|^2 ds \\ &\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned}$$

by taking stochastic integration. This completes the proof.

For the simplest case that $d = 1$, we have more consequences from the characterisation theorem. In this case we have

$$\gamma(t,x) = -\frac{b(t,x)}{\sigma(t,x)}$$

since $\sigma(t,x) \neq 0$. Set

$$u(t,x) := \frac{b(t,x)}{\sigma^2(t,x)} = -\frac{\gamma(t,x)}{\sigma(t,x)}, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

With the assumption on γ for the Girsanov theorem, we can rephrase our previous theorem in a slightly more concise manner

Theorem 1 in one dimension case

Let $v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$v(t, X_t) = v(0, X_0) - \int_0^t \frac{b(s, X_s)}{\sigma(s, X_s)} dB_s - \frac{1}{2} \int_0^t \left|\frac{b(s, X_s)}{\sigma(s, X_s)}\right|^2 ds$$

iff $u(t, x) := \frac{\partial}{\partial x} v(t, x)$ satisfies the following nonlinear PDE

$$\frac{\partial}{\partial t} u = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} u - \sigma \left(\frac{\partial}{\partial x} \sigma + \sigma u \right) \frac{\partial}{\partial x} u - \sigma u^2 \frac{\partial}{\partial x} \sigma \, .$$

Theorem 2

Let $v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$v(t, X_t) = v(0, X_0) - \int_0^t \frac{b(s, X_s)}{\sigma(s, X_s)} dB_s - \frac{1}{2} \int_0^t \left| \frac{b(s, X_s)}{\sigma(s, X_s)} \right|^2 ds$$

iff there exists a C^1-function $\Phi : \mathbb{R} \to \mathbb{R}$ such that for $u := \frac{\partial}{\partial x} v$

$$b(t, x) = \Phi(u(t, x)), \quad (t, x) \in [0, \infty) \times \mathbb{R}$$

and u satisfies the following (time-reversed) generalized Burgers equation

$$\frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi_1(u(t, x)) - \frac{1}{2} \frac{\partial}{\partial x} \Psi_2(u(t, x))$$

Theorem 2 (cont'd)

where

$$\Psi_1(r) := \int \frac{\Phi(r)}{r} dr, \quad \Psi_2(r) := r\Phi(r), \quad r \in \mathbb{R}.$$

The above generalized Burgers equation covers much more classes of specific nonlinear PDEs. Here we give three examples to explicate this point.

Example 1 Give a constant $\sigma > 0$. Let $b(t,x) = \sigma^2 u(t,x)$ and $\sigma(t,x) \equiv \sigma$, our SDE then becomes

$$dX_t = \sigma^2 u(t, X_t)dt + \sigma dB_t.$$

The C^1-function Φ is simply given by $\Phi(r) = \sigma^2 r$ and the corresponding PDE is a classical Burgers equation (time-reversed)

$$\frac{\partial}{\partial t}u(t,x) = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}u(t,x) - \sigma^2 u(t,x)\frac{\partial}{\partial x}u(t,x).$$

The next example shows that our generalized Burgers equation can be a porous media type PDE.

Example 2 We fix $m \in \mathbb{N}$. Let $b(t,x) = m[u(t,x)]^m$ and $\sigma(t,x) = \sqrt{m}[u(t,x)]^{\frac{m-1}{2}}$, our SDE then becomes

$$dX_t = m[u(t,X_t)]^m dt + \sqrt{m}[u(t,X_t)]^{\frac{m-1}{2}} dB_t.$$

The C^1-function Φ is then given by $\Phi(r) = mr^m$ and the corresponding PDE is a porous media type nonlinear PDE

$$\frac{\partial}{\partial t}u(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u^m(t,x) - m\frac{\partial}{\partial x}u^{m+1}(t,x).$$

The third example is to show that in the time-homogeneous case in the sense that b and σ are functions of the variable $x \in \mathbb{R}$ only, the corresponding PDE then determines a harmonic function.

Example 3 Let $b(t, x) = b(x)$ and $\sigma(t, x) = \sigma(x)$, our original SDE then reads as

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

and the corresponding PDE is a second order elliptic equation for harmonic functions

$$\frac{\partial^2}{\partial x^2}\Psi_1(u(x)) + \frac{\partial}{\partial x}\Psi_2(u(x)) = 0$$

where

$$\Psi_1(r) = \int \frac{\Phi(r)}{r}dr, \quad \Psi_2(r) = r\Phi(r), \quad r \in \mathbb{R}.$$

Here we'd like to extend our theorem to the case of SDEs on a general connected complete differential manifold. To this end, we need a proper framework to start with. Let us start with the following observation. In the situation of the SDEs on $\mathbb{R}^d$, if $X = (X_t)_{t\in[0,\infty)}$ solve

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

then, via martingale problem, the diffusion process X is associated with the Markov generator

$$L_t f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b^i(t,x)\frac{\partial f(x)}{\partial x_i}, \quad f \in C^2(\mathbb{R}^d)$$

with $a(t,x) := \sigma(t,x)\sigma^*(t,x)$.So let

$$g_t = (g_t^{ij}(\cdot)) := (\sigma\sigma^*)^{-1}(t,\cdot).$$

Then we have a time-dependent metric on $\mathbb{R}^d$ defined as follows

$$\langle x, y \rangle_{g_t} := \sum_{i,j=1}^{d} g_t^{ij} x_i y_j = \langle g_t x, y \rangle, \quad x, y \in \mathbb{R}^d.$$

Let ∇_{g_t} and Δ_{g_t} be the associated gradient and Laplacian, respectively. Then the generator for X can be reformulated as follows (cf. e.g. the classic books by D. Elworthy or by N. Ikeda and S. Watanabe)

$$L_t f = \frac{1}{2} \Delta_{g_t} f + \langle \tilde{b}(t, \cdot), \nabla_{g_t} f \rangle_{g_t}$$

for some smooth function $\tilde{b} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$. From this point of view, we intend to extend our theorem to a general connected complete differential manifold.

Let M be a d-dimensional connected complete differential manifold with a family of Riemannian metrics $\{g_t\}_{t\in[0,\infty)}$, which is smooth in $t \in [0,\infty)$. Clearly (M, g_t) is a Riemannian manifold for each $t \in [0,\infty)$. Let $\{b(t,\cdot)\}_{t\in[0,\infty)}$ be a family of smooth vector fields on M which is smooth in t as well. Let ∇_{g_t} and Δ_{g_t} denote the gradient and Laplacian operators induced by the metric g_t, respectively. Then the diffusion process X on M generated by the operator

$$L_t := \frac{1}{2}\Delta_{g_t} + b(t,\cdot)$$

can be constructed by solving the following SDE on M

$$dX_t = b(t, X_t)dt + \Phi_t \circ dB_t$$

where $\{B_t\}_{t\in[0,\infty)}$ is the d-dimensional Brownian motion, $\circ d$ stands for the Stratonovich differential, and Φ_t is the horizontal lift of X_t onto the frame bundle $O_t(M)$ of the Riemannian manifold (M, g_t), namely, Φ_t solves the following equation

$$d\Phi_t = H_{t,\Phi_t} \circ dX_t - \frac{1}{2}\Big\{ \sum_{i,j=1}^{d} (\partial_t g_t)(\Phi_t e_i, \Phi_t e_j) V_{ij}(\Phi_t) \Big\} d\,t,$$

where $H_{t,\cdot} : T(M) \to O_t(M)$ is the horizontal lift w.r.t. the metric g_t, $\{e_i\}_{1\le i\le d}$ is the canonical basis on $\mathbb{R}^d$ and $\{V_{ij}\}_{1\le i,j\le d}$ is the canonical basis of vertical vector fields, and $T(M)$ denotes the tangent bundle of M (cf. M. Arnaudon, K. Abdoulaye and A. Thalmaier, *C. R. Acad. Sci. Paris Ser. I* **346** (2008)). The next result is an extension of our characterisation theorem to M.

Theorem

Let $v : [0, \infty) \times M \to \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$v(t, X_t) = v(0, X_0) + \frac{1}{2} \int_0^t |b(s, X_s)|^2_{g_t} ds + \int_0^t \langle (\Phi_s^{-1} b(s, X_s), \circ dB_s \rangle_{g_t}$$

holds if and only if

$$b(t, x) = (\nabla_{g_t} v)(t, x), \quad (t, x) \in [0, \infty) \times M$$

and the following time-reversed Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} \left[(\Delta_{g_t} v)(t, x) + |\nabla_{g_t} v|^2_{g_t}(t, x) \right]$$

hold, where $|z|^2_{g_t} := \langle z, z \rangle_{g_t}$ for any vector z on M.

Proof Start from

$$v(t, X_t) = v(0, X_0) + \frac{1}{2}\int_0^t |b(s, X_s)|_{g_t}^2 ds + \int_0^t \langle (\Phi_s^{-1} b(s, X_s), \circ dB_s\rangle_{g_t}$$

we have

$$dv(t, X_t) = \frac{1}{2}|b(t, X_t)|_{g_t}^2 dt + \langle \Phi_t^{-1} b(t, X_t), \circ dB_t\rangle_{g_t} .$$

On the other hand, by the original SDE and Itô's formula, we get

$$dv(t, X_t) = \langle \Phi_t^{-1} \nabla_{g_t} v(t, X_t), \circ dB_t\rangle_{g_t} + \left\{ \frac{1}{2}\Delta_{g_t} v + \langle b, \nabla_{g_t} v\rangle_{g_t} \right\} (t, X_t) dt.$$

Now combining these two, we arrive the following

$$\nabla_{g_t} v(t, X_t) = b(t, X_t)$$

and

$$\left\{ \frac{1}{2} \Delta_{g_t} v + \langle b, \nabla_{g_t} v \rangle_{g_t} \right\} (t, X_t) = \frac{1}{2} |b(t, X_t)|^2_{g_t} .$$

Since $\{X_t\}_{t \in [0,\infty)}$ is supported by the whole manifold, the above two equalities imply

$$b(t, x) = (\nabla_{g_t} v)(t, x), \quad (t, x) \in [0, \infty) \times M$$

and

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} \left[(\Delta_{g_t} v)(t, x) + |\nabla_{g_t} v|^2_{g_t} (t, x) \right]$$

respectively. Reverse the derivation then completes the proof.

Remark

(i) An interesting question is to extend the argument here to infinite-dimensional SDEs, say, on Hilbert or Banach spaces or more generally on Itô's multi-Hilbertian spaces (cf K. Itô's SIAM Lecture Notes, 1984). The key point is to have Itô's formula for certain appropriate functionals.

(ii) The time-reversed Burgers-KPZ type equation we obtained on $\mathbb{R}^d$ is also linked to the stochastic Hamilton-Jacobi-Bellman equation. It is therefore an interesting question to see if one can recover a fuller picture of the mathematical physics of the stochastic HJB equations by exploiting the argument we have developed here.

Thank You!