Branching structure for an (L-1) random walk in random environment and its applications

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Contents

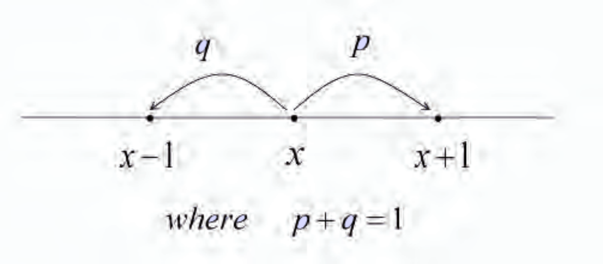
Nearest neighbor setting

(1) **Simple random walk:**

$$S_n = \sum_{i=1}^{n} \xi_i,$$

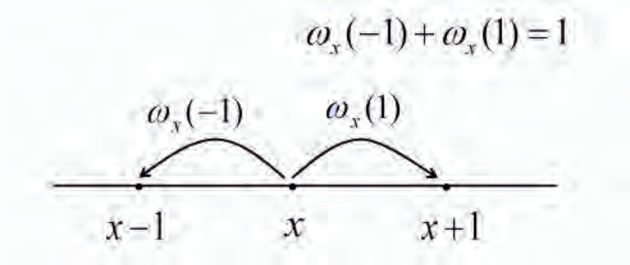
where ξ_i i.i.d., satisfying $P(\xi_i = 1) = p, P(\xi_i = -1) = q$.

$$P(S_n = j \big| S_{n-1} = i) = \begin{cases} p & \text{if } j = i + 1, \\ q & \text{if } j = i - 1. \end{cases}$$



(2) Simple RW with nonhomogeneous environment

$$P_\omega(X_{n+1} = x + 1 \big| X_n = x) = \omega_x(1);$$
$$P_\omega(X_{n+1} = x - 1 \big| X_n = x) = \omega_x(-1). \qquad (1)$$

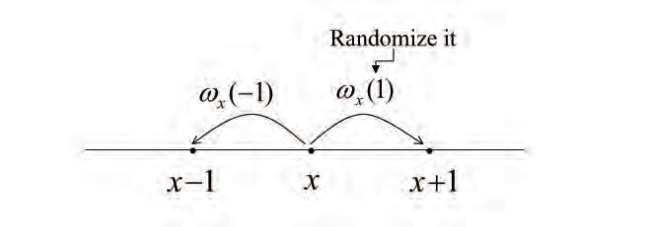


Here X_n cannot be written as sums of i.i.d. random variables any longer.

Random Walks in Random Environment

Randomizing the environment$\{\omega_x\}_{x\in\mathbb{Z}}$, letting $\{\omega_x\}_{x\in\mathbb{Z}} \sim \mathbb{P}$, one has

(3) Nearest neighbor Random Walk in Random Environment (RWRE)



$$\begin{aligned}\text{For fixing } \omega,\ & P_\omega(X_{n+1} = x+1 \big| X_n = x) = \omega_x(1);\\ & P_\omega(X_{n+1} = x-1 \big| X_n = x) = \omega_x(-1).\end{aligned} \quad (2)$$

$P^o_\omega(\cdot)$ is called quenched probability;

$P^o(\cdot) := \int P^o_\omega(\cdot)\mathbb{P}(d\omega)$ is called annealed probability.

Nearest neighbor RWRE

The history of the nearest neighbor RWRE:

(1) 0-1 law and LLN (ω_i i.i.d.) (Solomon 1975 Ann. Prob.)

(2) 0-1 law, LLN, CLT (ergodic ω)
(Alili 1999 JAP, Zeitouni 2004 LNM)

Let $\rho_i = \frac{\omega_i(-1)}{\omega_i(1)}$.

$\mathbb{E}(\log \rho_0)$ determines the recurrence and transience.

$\mathbb{E}(\rho_0)$ determines the LLN.

(3) LDP (i.i.d. ω)(See Greven-Hollander 1994 Ann. Prob.).

(4) LDP (ergodic ω)
(See Comets-Gantert-Zeitouni 2000 PTRF).

(5) Kesten-Kozlov-Spitzer [6](1975) proved a stable limit theorem for the nearest neighbor RWRE.

Difficulties arise when randomizing ω

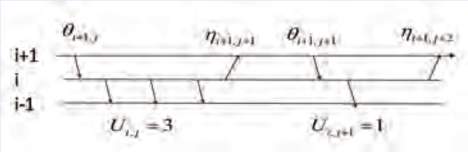
(1) Under the quenched probability P_ω^o, X_n cannot be written as sums of i.i.d. random variables any longer.
Simple random walk: $S_n = \sum_{i=1}^n \xi_i$, ξ_i i.i.d..

(2) Under the annealed probability P^o
$\{X_n\}$ is not a Markov chain.

One needs new tools for further studies of RWRE.

"Branching Structure": a powerful tool.

Branching structure for nearest neighbor RWRE



Define $\quad T_n = \inf[k > 0 : X_k = n]$.
While $X_n \to \infty$, define

$$U_i^n = \#\{k \le T_n : X_{k-1} = i, X_k = i-1\}.$$

$$P_\omega(U_i^n = k \big| U_{i+1}^n = 1) = [\omega_i(-1)]^k \omega_i(1).$$

Then

$$U_n^n = 0, U_{n-1}^n, \cdots, U_1^n$$

are the first n generations of a branching process with immigration in random environment. Moreover,

$$T_n = n + 2 \sum_{i \le n} U_i^n.$$

The function of the branching structure

The function of the branching structure:

1. One can count exactly the steps of the walk before T_n, $T_n = n + 2\sum_{i\leq n} U_i^n$.
2. Give the explicit expression of "invariant density", the key element of the method "The environment viewed from particle" (Zeitouni (2004) LNM).
3. Kesten-Kozlov-Spitzer (1975) proved the stable law for nearest neighbor RWRE.
4. Kesten (1977 Proceedings) proved the renewal theorem for nearest neighbor RWRE.
5. Gantert-Shi (2002 SPA) proved a limit theorem for the maximum of the occupation time.

Where this paper begins

Kesten-Kozlov-Spitzer (1975) (Branching structure for nearest neighbor RWRE)

In the literature we are aware of there is no description of the branching structure for non-nearest neighbor RWRE.

Questions:

(1) Are there any intrinsic connections between non-nearest neighbor RWRE and Branching Process in Random Environment?

(2) How to construct the branching structure for non-nearest neighbor RWRE? To what extent can we use the branching structure to study the limiting behaviors of non-nearest neighbor RWRE?

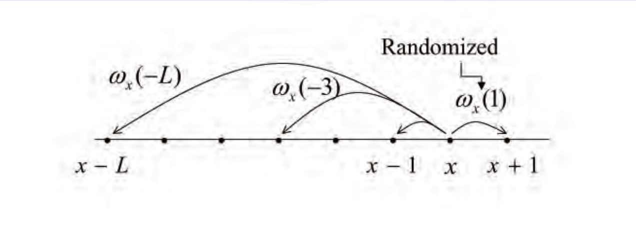
(L-1) RWRE

The model of the paper

(L-1) RWRE X_n:

$$P_\omega^o(X_{n+1} = x + l \big| X_n = x) = \omega_x(l),$$

$$\{\omega_x\}_{x\in\mathbb{Z}} = \{(\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1))\}_{x\in\mathbb{Z}} \sim \mathbb{P}.$$



The model

$\Lambda = \{-L, ..., 1\}/\{0\}$;
Σ : The simplex in $\mathbb{R}^{L+1}$.
$\Omega := \Sigma^{\mathbb{Z}}$, space of the environment.
μ : a probability measure on Σ.
$\mathbb{P} = \bigotimes^{\mathbb{Z}} \mu$ is the product measure on Ω, which makes $\omega_x, x \in \mathbb{Z}$ i.i.d..
Assume that $\mathbb{P}$ satisfies some elliptic condition: There exists $\varepsilon > 0$, such that

$$\mathbb{P}\left(\omega_0(-l)/\omega_0(1) \geq \varepsilon,\ \forall\ l \in \Lambda\right) = 1. \tag{3}$$

For $\omega \in \Omega$, define (L-1) RWRE as a Markov chain, with initial value $X_0 = 0$, transition probabilities

$$P_\omega(X_{n+1} = x + z | X_n = x) = \omega_x(z), \forall\ x \in \mathbb{Z}, z \in \Lambda. \tag{4}$$

$P_\omega^o(\cdot)$: is called the quenched probability;
$P^o(\cdot) := \int_\Omega P_\omega^o(\cdot)\mathbb{P}(d\omega)$: is called the annealed probability.

(L-1) RWRE

(L-R) RWRE was first introduced in Key (1984) Ann. Prob..

Brémont (2002) Ann. Prob. gave the 0-1 law and LLN.

Let

$$\overline{M}_i = \begin{pmatrix} a_i(1) & \cdots & a_i(L-1) & a_i(L) \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{5}$$

where $a_i(l) = \frac{\omega_i(-l)+\ldots+\omega_i(-L)}{\omega_i(1)}$.

Let γ_L be the great Lyapunov exponent of $\{\overline{M}_i\}$.

Relations between certain matrices

Define

$$M_{-i} = \begin{pmatrix} b_{-i}(1) & \cdots & b_{-i}(L-1) & b_{-i}(L) \\ 1 + b_{-i}(1) & \cdots & b_{-i}(L-1) & b_{-i}(L) \\ \vdots & \ddots & \vdots & \vdots \\ b_{-i}(1) & \cdots & 1 + b_{-i}(L-1) & b_{-i}(L) \end{pmatrix}, \tag{6}$$

where $b_i(l) = \frac{\omega_i(-l)}{\omega_i(1)}, 1 \leq l \leq L$;

$$B = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{7}$$

Then $\overline{M}_i = B^{-1} M_i B$. Moreover, $\{\overline{M_i}\}$ and $\{M_i\}$ share the same great Lyapunov exponent γ_L.

(L-1) RWRE

Theorem (Brémont 2002 Ann. Prob.)

(i) $\gamma_L < 0 \Rightarrow X_n \to \infty \ P^o\text{-}a.s.;$

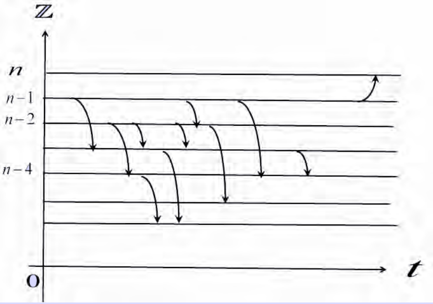
(ii) $\gamma_L > 0 \Rightarrow X_n \to -\infty \ P^o\text{-}a.s.;$

(iii) $\gamma_L = 0 \Rightarrow -\infty = \liminf X_n < \limsup X_n = \infty \ P^o\text{-}a.s..$

Also, Brémont showed a LLN under a condition called (IM) condition by him, that is,

$$\lim_{n\to\infty} \frac{X_n}{n} \to v \ P^o\text{-a.s..}$$

Branching structure for (L-1) RWRE($\limsup_{n\to\infty} X_n = \infty$)



$$U_{i,l}^n = \#\{0 < k < T_n : X_{k-1} > i, X_k = i - l + 1\},$$

$$U_i^n := (U_{i,1}^n, U_{i,2}^n, \cdots, U_{i,L}^n),$$

$$T_n = n + \sum_{i=-\infty}^{n-1} |U_i^n| + \sum_{i=-\infty}^{n-1} U_{i,1}^n = n + \sum_{i=-\infty}^{n-1} U_i^n (2, 1, ..., 1)^T.$$

Path decomposition and immigration structure

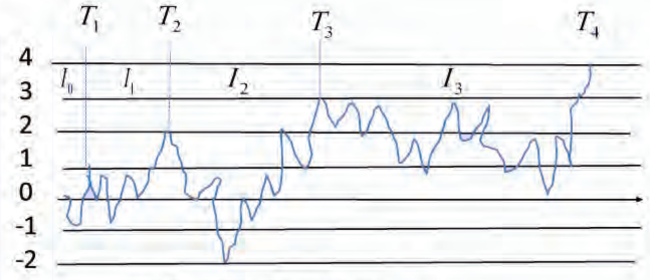


Figure . Immigration structures

This figure illustrates each of I_k corresponds to an immigrant and all immigration structures are independent by Markov property

Path decomposition

$$I_k =: \{X_m : T_k \le m < T_{k+1}\},\ k = 0, 1, ..., n-1.$$

For $1 \le l \le L$, $i < k$, define

$$U_l^n(k,i) = \#\{T_k \le m < T_{k+1} : X_{m-1} > i, X_m = i - l + 1\}.$$

Then $U_l^n(k,i)$ records all steps by the walk from above i downwards to $i - l + 1$ in the random walk path I_k. Let

$$U^n(k,i) = (U_1^n(k,i), U_2^n(k,i), ..., U_L^n(k,i)),$$

recording all steps by the walk crossing or reaching i from above i downwards in the random walk path I_k. For $i > k$, let $U^n(k,i) = \mathbf{0}$, and let $U^n(k,k) = e_1$. Then by definition of U_i^n and $U^n(k,i)$,

$$U_i^n = \sum_{k=(i+1)\vee 0}^{n-1} U^n(k,i).$$

Path decomposition

The offspring distribution of the type-1 particles

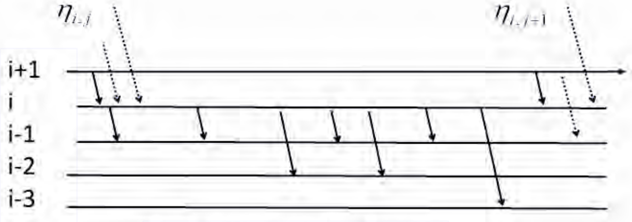


Figure 1. Children of a type-1 particle

This figure illustrates the children born to a type-1 particle. It has four type-1 children, two type-2 children, and one type-3 child.

$$P_\omega(U^n(k,i-1)=(u_1,...,u_L)\big|U^n(k,i)=e_1)$$
$$=\frac{(u_1+\cdots+u_L)!}{u_1!\cdots u_L!}\omega_i(-1)^{u_1}\cdots\omega_i(-L)^{u_L}\omega_i(1).$$

The offspring distribution of the type-$l(\geq 2)$ particles

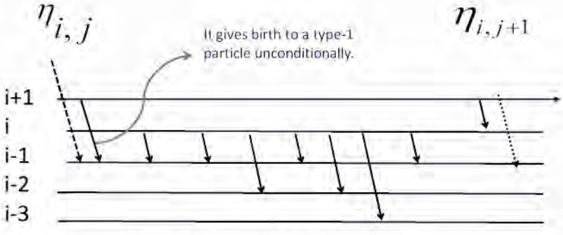


Figure 2. Children of a type-2 particle

Figure 2 illustrates the children born to a type-2 particle. The difference, compared with figure 1, is that before generating particles it gives birth to a type-1 particle with probability 1.

For $2 \leq l \leq L$,

$$P_\omega\left(U^n(k,i-1) = (u_1, ..., 1 + u_{l-1}, ..., u_L)\middle|U^n(k,i) = e_l\right)$$
$$= \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!}\omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L}\omega_i(1).$$

Branching structure for (L-1) RWRE

Theorem 1 (Branching structure for (L-1) RWRE)

Suppose that $\limsup_{n\to\infty} X_n = \infty$. Let $x_0 = (2, 1, ..., 1)$. Fix $n > 0$. Then

(a) $T_n = n + \sum_{i=-\infty}^{n-1} |U_i^n| + \sum_{i=-\infty}^{n-1} U_{i,1}^n = n + \sum_{i=-\infty}^{n-1} U_i^n x_0$;

(b) For $1 \le k \le n-1$, $U^n(k, *)$ is a nonhomogeneous multitype branching process starting at time k, with branching mechanism

$$P_\omega(U^n(k, i-1) = (u_1, ..., u_L) \big| U^n(k, i) = e_1) = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(1),$$

and for $2 \le l \le L$,

$$P_\omega \left(U^n(k, i-1) = (u_1, ..., 1 + u_{l-1}, ..., u_L) \big| U^n(k, i) = e_l \right) = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(1).$$

Branching structure

Theorem 1 (Continued))

further, conditioned on ω, $U^n(k,*)$, $k = 0, 1, ..., n-1$ are i.i.d. and particles of $U^n(k,*)$ have independent lines of offspring.

$$U^n_{n-1} = \mathbf{0}, U^n_{n-2}, \cdots, U^n_1, U^n_0, \quad (\text{Note: } U^n_i = \sum_{k=(i+1)\vee 0}^{n-1} U^n(k,i).)$$

are the first n generations of a multitype branching process with a type-1 immigration in each generation in random environment.

(c)

$$U^n_{n-1} = \mathbf{0}, U^n_{n-2}, \cdots, U^n_1, U^n_0$$

shares the same distribution with the first n generations of a MBPIRE $\{Z_{-n}\}_{n\geq 0}$, where the branching mechanism of $\{Z_{-n}\}_{n\geq 0}$ will be given in (8) and (9).

Multitype branching process with immigration in random environment

For $k \in \mathbb{Z}$ define $Z(k,m)$ to be an L-type branching process starting at time k in random environment ω. That is, given ω,

$$P_\omega(Z(k,m)=\mathbf{0})=1, \text{ if } m>k,$$
$$P_\omega(Z(k,k)=e_1)=1,$$

and for $m<k$

$$\begin{aligned} &P_\omega(Z(k,m)=(u_1,...,u_L)\big|Z(k,m+1)=e_1) \\ &=\frac{(u_1+\cdots+u_L)!}{u_1!\cdots u_L!}\omega_{m+1}(-1)^{u_1}\cdots\omega_{m+1}(-L)^{u_L}\omega_{m+1}(1), \end{aligned} \tag{8}$$

$$\begin{aligned} &P_\omega\left(Z(k,m)=(u_1,...,u_{l-2},u_{l-1}+1,u_l,...,u_L)\big|Z(k,m+1)=e_l\right) \\ &=\frac{(u_1+u_2+\cdots+u_L)!}{u_1!\cdots u_L!}\omega_{m+1}(-1)^{u_1}\cdots\omega_{m+1}(-L)^{u_L}\omega_{m+1}(1), \\ &\quad l=2,3,...,L. \end{aligned} \tag{9}$$

Multitype branching process with immigration in random environment

Let

$$Z_{-n} = \sum_{k=0}^{n-1} Z(-k,-n), \quad n > 0. \tag{10}$$

$\{Z_{-n}\}_{n\geq 0}$ is called Multitype Branching Process with Immigration in Random Environment (MBPIRE).

Given ω, let M_{-i} be such a matrix whose l-th row is the expected value of particles born to a type-l particle at time $-i$. Then

$$M_{-i} = \begin{pmatrix} b_{-i}(1) & \cdots & b_{-i}(L-1) & b_{-i}(L) \\ 1+b_{-i}(1) & \cdots & b_{-i}(L-1) & b_{-i}(L) \\ \vdots & \ddots & \vdots & \vdots \\ b_{-i}(1) & \cdots & 1+b_{-i}(L-1) & b_{-i}(L) \end{pmatrix}, \tag{11}$$

where $b_i(l) = \frac{\omega_i(-l)}{\omega_i(1)}, 1 \leq l \leq L$.

Ideas of the construction

Remark 1 (Ideas for the branching structure)

for (L-1) RWRE, the case becomes much more complicated because there are overlaps between different jumps. So to construct the branching structure, we use the following new ideas:

1) We treat the jumps reaching or crossing i downward from above i as the particles of i;

2) The types are determined by the position that the jumps reach. That is, if a jump crosses i downward from above i and reaches $i-l+1$ at last, the jump is a type-l particle of i;

3) We decompose the random walk path before T_n into n independent and non-intersection pieces and every piece corresponds to an immigration structure.

The first simple application

One follows from the branching structure immediately the following interesting result.

That is, for transient walk ($X_n \to \infty$), the tail of the minimum of the walk before hitting the positive real line decays faster than exponential rate.

Corollary 1

Suppose that $\gamma_L < 0$, and there is $q > 0$ such that $\mathbb{E}\|M_0\|^q < \infty$. Then for $t > 0$, one has that

$$P\left(\min_{0 \le n \le T_1} X_n < -t\right) < k_1 e^{-k_2 t}$$

where k_1 and k_2 are positive constants.

Testing of the branching structure

Consider a special (2-1) random walk. Fix $\omega := (..., \omega_0, \omega_0, \omega_0, ...) \in \Omega$, $\omega_0 = (q_1, q_2, p)$, where $p, q_1, q_2 \geq 0$, $p + q_1 + q_2 = 1$.
Suppose $\{X_n\}_{n\geq 0}$ is a random walk with $X_0 = 0$ and transition probabilities

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} p & \text{if } j = i+1, \\ q_1 & \text{if } j = i-1, \\ q_2 & \text{if } j = i-2. \end{cases}$$

Theorem 2 (The mean of T_1)

Suppose that $p > q_1 + 2q_2$. Then P_ω-a.s., $T_1 < \infty$ and $E_\omega(T_1) = \frac{1}{p - q_1 - 2q_2}$.

Proof. (Method 1: By Ward equation)

$$1 = E_\omega(X_{T_1}) = E_\omega(T_1)E_\omega(X_1) = (p - q_1 - 2q_2)E_\omega(T_1).$$

(Method 2: Using branching structure) $T_1 = 1 + \sum_{i=-\infty}^{0} U_i^1(2,1)^T$.

Stable Law for (L-1) RWRE

Let $\rho = \frac{1-\omega_0(1)}{\omega_0(1)}$.

Condition (C)

(C1) $\mathbb{E}\left(\log^+ \rho\right) < \infty$, where $\log^+ x := 0 \vee \log x$.

(C2) $\mathbb{P}\left(\rho > 1\right) > 0$.

Under **(C2)**, there exsits $\kappa_0 > 2$ such that

$$\mathbb{E}\Big[\Big(\min_{1\le i\le L}\{\sum_{j=1}^{L} M_0(i,j)\}\Big)^{\kappa_0}\Big] = \mathbb{E}(\rho^{\kappa_0}) > 1. \qquad (12)$$

Fixing κ_0 in (12), we give a new condition

(C3) $\mathbb{E}(\rho^{\kappa_0}\log^+\rho) < \infty$.

Let ϱ be the greatest eigenvalue of M_0. Then $\varrho > 0$.

(C4) The group generated by supp$[\log\varrho]$, is dense in $\mathbb{R}$. $\square$

Kesten 1973 proved that if $\gamma_L < 0$, then there is unique $\kappa \in (0, \kappa_0]$ such that

$$\log\rho(\kappa) := \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left(\| M_0 M_{-1}\cdots M_{-n+1} \|^{\kappa}\right) = 0. \qquad (13)$$

A Stable Law for (L-1) RWRE

Theorem 3 (Stable law for (L-1) RWRE)

Suppose that Condition **(C)** holds and $\gamma_L < 0$. Fix κ in (13).
Let $L_\kappa(x)$ be a κ-stable law (if $\kappa < 1$, L_κ has support $[0, \infty)$; if $\kappa > 1$, L_κ has zero mean).
(i) if $0 < \kappa < 1$, then

$$\lim_{n\to\infty} P(n^{-\frac{1}{\kappa}} T_n \le x) = L_\kappa(x),$$

$$\lim_{n\to\infty} P(n^{-\kappa} X_n \le x) = 1 - L_\kappa(x^{-\frac{1}{\kappa}});$$

(ii) if $\kappa = 1$, then for suitable $D(n) \sim \log n$ and $\delta(n) \sim (A_1 \log n)^{-1} n$,

$$\lim_{n\to\infty} P(n^{-1}(T_n - A_1 n D(n\mu^{-1})) \le x) = L_1(x),$$

$$\lim_{n\to\infty} P(n^{-1}(\log n)^2 (X_n - \delta(n)) \le x) = 1 - L_1(-A_1^2 x);$$

Stable Law

Theorem 3 (Stable law for (L-1) RWRE (Continued))

(iii) if $1 < \kappa < 2$, then

$$\lim_{n\to\infty} P\left(n^{-\frac{1}{\kappa}}(T_n - A_\kappa n) \le x\right) = L_\kappa(x),$$

$$\lim_{n\to\infty} P\left(n^{-\frac{1}{\kappa}}\left(X_n - \frac{n}{A_\kappa}\right) \le x\right) = 1 - L_\kappa(-xA_\kappa^{1+\kappa^{-1}});$$

(iv) if $\kappa = 2$, then

$$\lim_{n\to\infty} P\Big(\frac{T_n - A_2 n}{B_1\sqrt{n\log n}} \le x\Big) = \Phi(x),$$

$$\lim_{n\to\infty} P\left(A_2^{\frac{3}{2}} B_1^{-1} (n\log n)^{-\frac{1}{2}}\left(X_n - \frac{n}{A_2}\right) \le x\right) = \Phi(x);$$

Theorem 3 (Stable law for (L-1) RWRE (Continued))

(v) if $\kappa > 2$, then

$$\lim_{n\to\infty} P\Big(\frac{T_n - B_3 n}{B_2\sqrt{n}} \le x\Big) = \Phi(x),$$

$$\lim_{n\to\infty} P\left(B_3^{\frac{3}{2}} B_2^{-1} n^{-\frac{1}{2}}\left(X_n - \frac{n}{B_3}\right) \le x\right) = \Phi(x),$$

where $0 < A_\kappa, B_i < \infty$ are suitable constants, and $\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}ds.$

Ideas of the proof: $T_n = n + \sum_{i=-\infty}^{n-1} U_i^n x_0$. It suffices to show that $\sum_{i=0}^{n-1} U_i^n x_0$ converges to L_κ after suitable renormalization. But

$$\sum_{i=0}^{n-1} U_i^n x_0 \overset{\mathcal{D}}{=} \sum_{i=0}^{n-1} Z_{-i} x_0 \approx \sum_{k=1}^{n/E(\nu)} W_k x_0.$$

The slowdown properties of RWRE

From (i) of Theorem 3 (Stable Law) one has the following theorem, which tells that the walk slows down after randomizing the environment.

Theorem 4 (The slowdown properties of RWRE)

Suppose that Condition **(C)** holds, $\gamma_L < 0$, and that $0 < \kappa < 1$. Then for $\kappa < s < 1$ one has that

$$\lim_{n\to\infty} \frac{X_n}{n^s} = 0, \ P^o\text{-a.s.}.$$

For simple random walk(linear speed)

$$S_n \to \infty \Leftrightarrow \frac{S_n}{n} \to E(\xi_1) > 0.$$

But Theorem 4 reveals that for (L-1) RWRE(sub-linear speed),

$$\text{though } X_n \to \infty, \frac{X_n}{n^s} \to 0, \ \kappa < s.$$

LLN for (L-1) RWRE

For $n \geq 0$, define $\overline{\omega}(n) = \theta^{X_n}\omega$. Then $\{\overline{\omega}(n)\}$ is a Markov chain with transition kernel

$$\overline{P}(\omega, d\omega') = \omega_0(1)\delta_{\theta\omega=\omega'} + \sum_{l=1}^{L} \omega_0(-l)\delta_{\theta^{-l}\omega=\omega'}.$$

Define

$$\pi(\omega) := \frac{1}{\omega_0(1)}\left(1 + \sum_{i=1}^{\infty} e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T\right).$$

Let $\tilde{\pi}(\omega) = \frac{\pi(\omega)}{\mathbb{E}(\pi(\omega))}$. Then one has

LLN for (L-1) RWRE

Theorem 5 (LLN and Invariant Measure)

Suppose that $\mathbb{E}(\pi(\omega)) < \infty$. Then

(i) $\gamma_L < 0$;

(ii) $\tilde{\pi}(\omega)\mathbb{P}(d\omega)$ is invariant under kernel $\overline{P}(\omega, d\omega')$, that is

$$\int 1_B \tilde{\pi}(\omega)\mathbb{P}(d\omega) = \iint 1_{\omega' \in B} \overline{P}(\omega, d\omega')\tilde{\pi}(\omega)\mathbb{P}(d\omega), \; B \in \mathcal{F};$$

(iii) moreover $\mathbb{P}$-a.s., $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}(\pi(\omega))}$.

Remark 2

LLN was proved in Brémont (2002) where the author introduced a so-called (IM) Condition, and gave the Invariant Measure by studying its transition probability. By the branching structure we can calculate the Quenched Mean $E_\omega^o(T_1)$ to give the explicit expression of the Invariant Measure, and prove the LLN directly by a method known as "the environment viewed from particles".

The tail of the expected value of the total progeny of a immigration

Let $\{Z_{-n}\}_{n\geq 0}$ be the MBPIRE with negative time defined in (10). Let

$$Y_{-k} = \sum_{m=k+1}^{\infty} Z(-k, -m).$$

Note that for $m > k$, one has $E_\omega(Z(-k, -m)) = M_{-k}M_{-k-1}\cdots M_{-m+1}$. Then

$$\eta_{-k} := \sum_{m=k+1}^{\infty} M_{-k}...M_{-m+1} \tag{14}$$

the expected offspring matrix of Y_{-k}. We prove the follow theorem:

Total progeny of an immigration

Theorem 6 (Tail of total progeny of an immigration)

Suppose that Condition **(C)** holds and $\gamma_L < 0$. Then for κ in (13) and suitable constant $K_2 = K_2(x_0) \in (0, \infty)$, one has that

$$\lim_{t \to \infty} t^{\kappa} \mathbb{P}(x\eta_0 x_0 \geq t) = K_2 |xB|^{\kappa}, \tag{15}$$

where $x \in \mathbb{R}^L$ with all coordinates positive and $|x| > 0$.

Remark 3

Indeed, Kesten (1973) showed that

$$\lim_{t \to \infty} t^{\kappa} \mathbb{P}(x\eta_0 x_0 \geq t) = K(x, x_0),$$

where K depends on x and x_0. The main contribution of Theorem 6 is that it tells how the constant K depends on x. K_2 in (15) is independent of x.

Tail of the total population before regeneration

Let $\nu_0 \equiv 0$, and define recursively

$$\nu_n = \min\{m > \nu_{n-1} : Z_{-m} = \mathbf{0}\}, \ \forall \ n > 0.$$

Write ν_1 simply as ν.

ν is the regeneration time of $\{Z_{-n}\}_{n\geq 0}$;

From the point of view of the random walk ν is $\{X_n\}_{n\geq 0}$ the first position where the walk will never revisit after passing it.

Define

$$W = \sum_{k=0}^{\nu-1} Z_{-k}.$$

W is the total population born before regeneration time ν.

Tail of the total population before regeneration

Theorem 7 (Tail of W)

Suppose that Condition **(C)** holds and $\gamma_L < 0$. If $\kappa > 2$, $E((Wx_0)^2) < \infty$; if $\kappa \leq 2$, there exists $0 < K_3 < \infty$ such that

$$\lim_{t\to\infty} t^{\kappa} P(Wx_0 \geq t) = K_3.$$

Remark 4

Theorem 7 reveals that the total population of the MBPIRE before regeneration, Wx_0, belongs to the domain of attraction of a κ stable law. Moreover Theorem 7 is the key step for the proof of the stable limit theorem (Theorem 3.).

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非 常 感 谢

Thanks a lot!