Convergence rate for Markov transition matrices

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(based on a joint work with Yong-Hua Mao)

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Assumptions

Given a probability transition matrix $P = (p_{ij})$:

$$p_{ij} \geq 0, \quad \sum_j p_{ij} = 1.$$

Assume that P is ergodic:

- P is irreducible: $\forall i, j, \exists n, p_{ij}^{(n)} > 0$;
- P is aperiodic: $d = \text{gcd}\left\{n : p_{ii}^{(n)} > 0\right\} = 1$;
- P has a stationary distribution: $\exists$ a probability $\pi = (\pi_j)$ s.t. $\pi = \pi P \Longleftrightarrow \lim_{n\to\infty} p_{ij}^{(n)} = \pi_j, \forall i, j$.

Assume further:

- P is reversible w.r.t. $\pi : \pi_i p_{ij} = \pi_j p_{ji}$.

Problem

A basic problem is to study the convergence rate of

$$\sum_{j \in E} \left| p_{ij}^{(n)} - \pi_j \right| \to 0.$$

Setting-up

Let $L^2 = L^2(\pi)$ be the Hilbert space, then P is a self-adjoint operator in L^2 and the spectrum of P $\sigma(P) \subset [-1, 1]$. By using the spectral mapping theorem, we have

$$||P^n - \pi||_{L^2 \to L^2} \leq r^n,$$

where $r = r_1 \vee r_{-1}$,

$$r_1 := \sup \{x < 1 : x \in \sigma(P)\},$$

and

$$r_{-1} := -\inf \{x : x \in \sigma(P)\}.$$

Setting-up

Let $\widetilde{P}_t$ or $(\widetilde{X}_t)$ be a Markov jump process associated with $Q = P - I$. Define Dirichlet form:

$$D(f) = \frac{1}{2}\sum_{i,j} \pi_i q_{ij}(f_j - f_i)^2 = \frac{1}{2}\sum_{i,j} \pi_i p_{ij}(f_j - f_i)^2$$

and Poincaré variational formula:

$$\lambda_1 = \inf\left\{D(f) : \pi(f) = 0, \pi(f^2) = 1\right\}.$$

Then

$$||\widetilde{P}_t - \pi||_{L^2 \to L^2} \le \mathrm{e}^{-\lambda_1 t}.$$

Question & "Answer"

- A good relationship between λ_1 and r_1:

$$r_1 + \lambda_1 = 1.$$

- Q: if $r_1 < 1$ (or $\lambda_1 > 0$), then $r < 1$ or $r_{-1} < 1$?

- A: This is true under our assumptions!

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Contents

Main theorem

Theorem 1

If $r_1 < 1$, *then*

$$r_1 \leq r \leq \left(\frac{a\delta + b}{a+b}\right)^{\frac{1}{2}} \vee \left(\frac{4\xi}{1+3\xi}\right),$$

where

$$a := (3+5\xi)(1-\xi^2), \qquad b := 2(2\xi - r_1)(1+3\xi)^2,$$

$$\xi := 1 - \pi_0(1-r_1) < 1, \quad \delta := \sum_{n\geq 1} f_{00}^{(2n)} < 1.$$

δ is the probability that the chain comes back to state 0 firstly in even steps, starting from state 0.

Idea of the proof

The basic idea of the proof is based on the following theorem.

Theorem 2 (Mao.2010)

If there exists $\lambda > 1$ such that

$$\mathbb{E}_0 \lambda^{\tau_0^+} \leq M < \infty, \tag{1}$$

and let

$$\rho = \sup \left\{ s \leq \lambda : \sum_{n=1}^{\infty} s^{2n} f_{00}^{(2n)} < 1 \right\}, \tag{2}$$

then we have

$$r \leq \rho^{-1} < 1.$$

Idea of the proof

- Find a function of λ_1, say $\phi(\lambda_1)$, such that for any $1 < \lambda < \phi(\lambda_1)$,

$$\mathbb{E}_0 \lambda^{\tau_0^+} \leq M < \infty.$$

- Solve the inequality $\sum_{n=1}^{\infty} s^{2n} f_{00}^{(2n)} < 1$. Then we can find a function of λ, say $\psi(\lambda)$, such that

$$r \leq \psi(\lambda).$$

- Combine the above two estimations, we have

$$r \leq \inf\{\psi(\lambda) : 1 < \lambda < \phi(\lambda_1)\}.$$

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Moments in continuous time

- $\lambda_1 \geq \lambda_0 \geq \pi_0\lambda_1$, where Dirichlet spectral gap:

$$\lambda_0 = \inf\left\{D(f) : f_0 = 0, \pi(f^2) = 1\right\}.$$

Thus we have

$$\lambda_0 \geq \pi_0(1 - r_1) > 0.$$

Moments in continuous time

- $$||\widehat{P}_t||_{L^2 \to L^2} \le e^{-\lambda_0 t},$$

  where

  $\widetilde{\tau}_0 = \inf\left\{t \ge 0 : \widetilde{X}_t = 0\right\}, \widehat{p}_{ij}(t) = \mathbb{P}_i[\widetilde{X}_t = j, t < \widetilde{\tau}_0].$

  From this we have for any $\lambda < \lambda_0$ and $i \ge 1$,

  $$\mathbb{E}_i e^{\lambda \widetilde{\tau}_0} \le \frac{\lambda_0(1-\pi_0)}{\pi_i(\lambda_0 - \lambda)} < \infty.$$

  Thus the above inequality holds for $\lambda < \pi_0(1 - r_1)$.

Passing to moments in discrete time

- Let $\tau_0^+ = \inf\{n \geq 1 : X_n = 0\}$ be the return time, then

$$\mathbb{E}_i \left(\frac{1}{1-\lambda} \right)^{\tau_0^+} = \mathbb{E}_i e^{\lambda \widetilde{\tau}_0}, \quad \forall i \neq 0.$$

- By a theorem due to Cogburn(1975), we have for any $\lambda < \frac{1}{1-\pi_0\lambda_1}$,

$$\mathbb{E}_0 \lambda^{\tau_0^+} \leq \lambda + (1-\pi_0)\frac{\lambda_1\lambda(\lambda-1)}{1-\lambda(1-\pi_0\lambda_1)}.$$

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Solve inequality

Let $a_n := \sum_{k=n}^{\infty} \mathbb{P}_0[\tau_0^+ = 2k]$.

$$
\begin{aligned}
F_{00}^{(0)}(s) &= \sum_{n=1}^{\infty} s^{2n} f_{00}^{(2n)} = s^2 a_1 + \left(1 - \frac{1}{s^2}\right) \sum_{n=2}^{\infty} s^{2n} a_n \\
&\leq s^2\delta + \left(1 - \frac{1}{s^2}\right) \sum_{n=2}^{\infty} s^{2n} \mathbb{P}_0[\tau_0^+ \geq 2n] \\
&\leq s^2\delta + \left(1 - \frac{1}{s^2}\right) \mathbb{E}_0 \lambda^{\tau_0^+} \sum_{n=2}^{\infty} s^{2n} \lambda^{-2n} \\
&= s^2\delta + \frac{(\lambda+1)^2(s^2-1)}{\lambda^2(\lambda-1)(3\lambda+1)} M \\
&< 1 \,.
\end{aligned}
$$

Solving this inequality, we can prove theorem 1.

A typical example: Random walk

- Let $P = (p_{ij})$ on state space $E = \mathbb{Z}_+$ with $p_{i,\,i+1} = b_i > 0 (i \geq 0)$, $p_{ii} = c_i \geq 0 (i \geq 0)$, $p_{i,\,i-1} = a_i > 0 (i \geq 1)$.
- P is aperiodic iff $c_0 > 0$(say), and it is ergodic iff $\mu := \sum_{n=0}^{\infty} \mu_n < \infty$, where $\mu_0 = 1$, $\mu_n = b_0 b_1 \cdots b_{n-1} / a_1 a_2 \cdots a_n$. Let $\pi_i = \mu_i / \mu$, $i \geq 0$.

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A typical example: Random walk

- It is known that $r_1 < 1(\lambda_1 > 0)$ iff $\kappa < \infty$, where

$$\kappa = \sup_{n \geq 0} \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i} \sum_{i=n}^{\infty} \mu_i < \infty.$$

- Precisely, we know

$$(4\kappa)^{-1} \leq \lambda_1 \leq \mu \kappa^{-1},$$

or

$$1 - \mu\kappa^{-1} \leq r_1 \leq 1 - (4\kappa)^{-1}.$$

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Convergence rate for RW

- $$r \geq 1 - \mu\kappa^{-1}.$$

- $$r \leq \left(\frac{a\delta + b}{a+b}\right)^{\frac{1}{2}} \vee \frac{4\xi}{1+3\xi},$$

  where

  $$a = (3+5\xi)(1-\xi^2),\ b = 2(2\xi - r_1)(1+3\xi)^2,$$

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  $$\xi := 1-\pi_0(1-r_1) \leq 1-(4\mu\kappa)^{-1},\ \delta := \sum_{n\geq 1} f_{00}^{(2n)} \leq 1-c_0.$$

- $$r \geq 1 - \mu\kappa^{-1}.$$

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Strong ergodicity

- The Markov chain (X_n) or P is strongly ergodic, if there exist $\gamma < \infty$ and $\alpha = \alpha(\gamma) < 1$ such that

$$\sup_{i\in E} \|P_{i\cdot}^{(n)} - \pi\|_{\text{Var}} \le \gamma\alpha^n, \quad \forall n \ge 0. \tag{3}$$

- The Markov chain $(\widetilde{X}_t)$ or $Q = P - I$ is strongly ergodic, if there exist $\widetilde{\gamma} < \infty$ and $\widetilde{\alpha} = \widetilde{\alpha}(\widetilde{\gamma}) > 0$ such that

$$\sup_{i\in E} \|\widetilde{P}_{i\cdot}(t) - \pi\|_{\text{Var}} \le \widetilde{\gamma}\, \mathrm{e}^{-\widetilde{\alpha}t}, \quad \forall t \ge 0. \tag{4}$$

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Convergence rate for discrete time

Let

$$\beta^{-1} = \sum_{n\geq 1} f_{00}^{(2n-1)}, \quad M_1 = \frac{2}{\widetilde{\alpha}}\left[\frac{1}{\pi_0}\log\frac{\widetilde{\gamma}}{\pi_0}\right],$$

and

$$M = \frac{1}{2}(\beta + M_1 + \beta M_1).$$

Theorem 3

If $(\widetilde{X}_t)$ is strongly ergodic with convergence rate $\widetilde{\alpha}$, then there exist $C_1 < \infty$ and $C_2 = C_2(C_1, M) < \infty$ such that

$$\sup_{i\in E}\sum_{k\in E}|p_{ik}^{(n)} - \pi_k| \leq e^{-\frac{n}{2M}}\left[C_2 + \frac{\mathrm{e}^{1-\frac{1}{2M}}}{M}n + \frac{C_1\,\mathrm{e}^{1+\frac{1}{2M}}}{8M}n^2\right].$$

Renewal formula

Lemma 4

$$\sum_{k\in E}\left|p_{ik}^{(n)}-\pi_k\right| \le 2\mathbb{P}_i[\tau_0>n]+\sum_{m=1}^{n}\sum_{k\in E}\left|p_{0k}^{(n-m)}-\pi_k\right| f_{i0}^{(m)}.$$

- For $n \ge \widehat{M} := \sup_i \mathbb{E}_i\tau_0$, $\sup_i \mathbb{P}_i[\tau_0 \ge n] \le \frac{e\,n}{\widehat{M}}\,e^{-\frac{n}{\widehat{M}}}$.
- $\sup_i \mathbb{E}_i\tau_0 = \sup_i \mathbb{E}_i\widetilde{\tau}_0$.

Theorem 5 (Mao,2006)

If $(\widetilde{X}_t)$ is strongly ergodic with convergence rate $\widetilde{\alpha}$, then

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Convergence rate for continuous time

Let

$$C_0 = (\pi_0^{-1} - 1)^{\frac{1}{2}}, \quad M_0 = \frac{2 \log \frac{\pi_0}{\gamma}}{\pi_0 \log \alpha}.$$

Theorem 6

If (X_n) is strongly ergodic with convergence rate α, then there exists $C_0 < \infty$ such that

$$\sup_{i \in E} \sum_{k \in E} |p_{ik}(t) - \pi_k| \leq \mathrm{e}^{-\frac{t}{M_0}} \left[\frac{C_0\,\mathrm{e}}{2} + \frac{(2 - C_0)^+\,\mathrm{e}}{M_0} t + \frac{C_0\,\mathrm{e}}{2M_0^2} t^2 \right].$$

Renewal formula

Lemma 7

$$\sum_{k\in E}|p_{ik}(t)-\pi_k| \le 2\mathbb{P}_i[\widetilde{\tau}_0 > t] + \int_0^t \sum_{k\in E}|p_{0k}(t-s)-\pi_k|\,\mathrm{d}\mathbb{P}_i(\widetilde{\tau}_0 \le s).$$

- For $t \ge \widehat{M} := \sup_i \mathbb{E}_i\widetilde{\tau}_0$, $\sup_i \mathbb{P}_i[\widetilde{\tau}_0 \ge t] \le \frac{\mathrm{e}\,t}{\widehat{M}}\,\mathrm{e}^{-\frac{t}{\widehat{M}}}$.
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- For $t \ge \widehat{M} := \sup_i \mathbb{E}_i \widetilde{\tau}_0$, $\quad \sup_i \mathbb{P}_i[\widetilde{\tau}_0 \ge t] \le \frac{\mathrm{e}\, t}{\widehat{M}} \mathrm{e}^{-\frac{t}{\widehat{M}}}$.
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Absorbing process

- Recall the Dirichlet spectral gap:

$$\lambda_0 = \inf \left\{ D(f) : f_0 = 0, \pi(f^2) = 1 \right\}.$$

- Suppose state 0 is the absorbing point of (X_n). Then P transforms to P_D, which P_D is the matrix obtained from P by deleting the row and column corresponding to 0. Define $r = r_1 \vee r_{-1}$, where

$$r_1 := \sup \{x : x \in \sigma(P_D)\},$$

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Absorbing process

Theorem 9

When P is reversible with respect to π, and has an absorbing point 0, then we have $r = r_1$.

- $$\mathbb{E}_i\left(\frac{1}{1-\lambda}\right)^{\tau_0^+} = \mathbb{E}_i e^{\lambda\tilde{\tau}_0}, \quad \forall i \neq 0.$$
- By re-normalizing method due to Feng-Yu Wang (2000), or a proposition of Sokal and Thomas (1989).

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Transient case

When P is transient with a symmetric measure μ, define Dirichlet form:

$$D(f) := \frac{1}{2} \sum_{i,j} \mu_i p_{ij} (f_i - f_j)^2,$$

and Dirichlet spectral gap:

$$\overline{\lambda} := \inf\{D(f) : \mu(f^2) = 1, f \in \mathcal{K}\},$$

where $\mathcal{K}$ is the set of functions with finite support.

Similarly,

$$r_1 + \overline{\lambda} = 1.$$

Approximating method

Theorem 10

When P is transient with a symmetric measure μ, then $r = r_1$.

- For fixed $n \in \mathbb{N}$, let P_{D_n} be the matrix obtained from P by deleting the rows and columns corresponding to the states which are larger than n. Then

$$r(P_{D_n}) = 1 - \lambda_0^{(n)}, \tag{5}$$

where

$$r(P_{D_n}) := \sup \{|\lambda| : \lambda \in \sigma(P_{D_n})\},$$

$$\lambda_0^{(n)} := \inf\{D(f) : \mu(f^2) = 1, f_i = 0, \forall i > n\}.$$

- It follows by letting $n \to \infty$ in (5) that $r = r_1$.

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Theorem 10

When P is transient with a symmetric measure μ, then $r = r_1$.

- For fixed $n \in \mathbb{N}$, let P_{D_n} be the matrix obtained from P by deleting the rows and columns corresponding to the states which are larger than n. Then

$$r(P_{D_n}) = 1 - \lambda_0^{(n)}, \tag{5}$$

where

$$r(P_{D_n}) := \sup\left\{|\lambda| : \lambda \in \sigma(P_{D_n})\right\},$$
$$\lambda_0^{(n)} := \inf\{D(f) : \mu(f^2) = 1, f_i = 0, \forall i > n\}.$$

- It follows by letting $n \to \infty$ in (5) that $r = r_1$.

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QED

Thank You