Potential Theory of Subordinate Brownian Motions

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References

This talk is based on the following joint papers with Panki Kim and Zoran Vondracek:

[1]. Boundary Harnack principle for subordinate Brownian motion. *Stoch. Proc. Appl.*, **119** (2009), 1601–1631.

[2]. Two-sided Green function estimates for killed subordinate Brownian motions, preprint, 2010

References (Cont)

For background materials on subordinate Brownian motions, see Chapter 5 of:

[3]. K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R, Song and Z. Vondracek, *Potential analysis of stable processes and its extensions*, Lecture Notes in Mathematics, Vol. 1980. Springer-Verlag, Berlin, 2009.

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Background and Motivation

Many physical and economic systems should be and in fact have been successfully modeled by discontinuous Markov processes. Discontinuous Markov processes are also very important from a theoretical point of view.

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Due to their importance both in theory and in applications, discontinuous Markov processes, particularly Lévy processes, have been receiving intensive study in recent years.

The recent study on discontinuous Markov processes started in the 1990's with the study of symmetric α-stable processes, $\alpha \in (0,2)$. Recall that a symmetric α-stable process is a Lévy process with Lévy exponnet $\Phi(\theta) = |\theta|^\alpha$, that is,

$$\mathbb{E}[\exp\{i\theta \cdot X_t\}] = \exp\{-t|\theta|^\alpha\}, \quad \theta \in \mathbb{R}^d.$$

One of the first result in this respect is the following sharp Green function estimates of symmetric α-stable processes in bounded $C^{1,1}$ domains, obtained in dependently by Chen-Song and Kulczycki.

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Theorem

Suppose that $d \geq 2$ and $\alpha \in (0,2)$. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and let G_D be the Green function of the symmetric α-stable process in D. Then

$$G_D(x,y) \asymp \left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|^{\alpha}}\right) \frac{1}{|x-y|^{d-\alpha}}, \quad x,y \in D.$$

The same form of the estimates in the case $\alpha = 2$ (and $d \geq 3$) were obtained much earlier by Widman and Zhao.

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In 2002, the above Green function estimates were generalized to relativistic stable processes by Ryznar. Recall that a relativistic α-stable process (with relativistic mass $m > 0$) is a Lévy process with Lévy exponent $\Phi(\theta) = (|\theta|^2 + m^{2/\alpha})^{\alpha/2} - m$. Then in 2003, Chen-Song developed a general perturbation result which includes Ryznar's result as special cases.

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Suppose that $d \geq 2$, $\alpha \in (0, 2)$ and $m > 0$. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and let G_D be the Green function of the relativistic α-stable process with mass m in D. Then

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In a recent paper Chen-Kim-Song, sharp estimates on the heat kernel of the Lévy process with Lévy exponent $\Phi(\theta) = |\theta|^\alpha + |\theta|^\beta$, $0 < \beta < \alpha < 2$, were obtained. As a consequence of the heat kernel estimates, one can easily get the following Green function estimates which can not be obtained by perturbation methods.

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Suppose that $d \geq 2$, $0 < \beta < \alpha < 2$. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and let G_D be the Green function in D of the Lévy process with Lévy exponent $\Phi(\theta) = |\theta|^\alpha + |\theta|^\beta$, $0 < \beta < \alpha < 2$. Then

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The common feature of those Green function estimates is that both the distance between the points, $|x - y|$, and distances to D^c, $\delta_D(x), \delta_D(y)$, appear as arguments of the *power functions*.

However, it follows from Chapter 5 of LNM Vol 1980 that the asymptotic behavior the Green function $G(x, y)$ of many transient symmetric Lévy processes are of the form

$$G(x, y) \sim \frac{1}{|x - y|^{\alpha - d} \ell(|x|^{-2})} \quad \text{as } |x| \to 0$$

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Outline

A C^∞ function $\phi : (0,\infty) \to [0,\infty)$ is called a Bernstein function if $(-1)^n D^n \phi \le 0$ for every positive integer $n \ge 1$.

Every Bernstein function has a representation

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\,\mu(dt)$$

where $a, b \ge 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\,\mu(dt) < \infty$.

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A Bernstein function ϕ is called a complete Bernstein function if the Lévy measure μ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \geq 0$ for every non-negative integer $n \geq 0$. We will denote the Lévy density by $\mu(t)$.

By Bernstein's theorem, function on $(0, \infty)$ is a completely monotone function iff it is the Laplace transform of a measure on $[0, \infty)$.

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A subordinator is a just a nonnegative Lévy process. A subordinator $S = (S_t : t \geq 0)$ is usually characterized by its Laplace exponent ϕ, that is

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Let U be the renewal function of the subordinator S, that is, for any $t > 0$,

$$U(t) = \mathbb{E} \int_0^\infty 1_{\{X_s \leq t\}} ds.$$

S is a complete subordinator iff U is absolutely continuous with respect to the Lebesgue measure and the density (also called the potential density of S) is completely monotone.

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Let $W = (W_t = (W_t^1, \ldots, W_t^d) : t \geq 0)$ be a d-dimensional Brownian motion, and let $S = (S_t : t \geq 0)$ be an independent subordinator. The process $X = (X_t : t \geq 0)$ defined by $X_t := W_{S_t}$, $t \geq 0$ is called a subordinate Brownian motion.

If the Laplace exponent of S is ϕ, then the Lévy exponent of the subordinate Brownian motion X is given by $\Phi(\theta) = \phi(|\theta|^2)$. By choosing the Laplace exponent $\phi(\lambda)$ as $\lambda^{\alpha/2}$, $(\lambda + m^{2/\alpha})^{\alpha/2} - m$ and $\lambda^{\beta/2} + \lambda^{\alpha/2}$ respectively, the resulting subordinate Brownian motion turns out to be a symmetric α-stable process, a relativistic stable process and an independent sum of β and α-stable processes respectively.

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In this talk, I will always assume that S is a complete subordinator with Laplace exponent ϕ. Our standing assumption is that

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda),$$

where ℓ is a slowly varying function at infinity, $0 < \alpha < 2$. This is just an assumption on the asymptotic behavior of ϕ at infinity.

It is easy to check that, when $d \geq 3$, the subordinate Brownian motion is transient. And we will use $G(x, y)$ to denote the Green function of X. When $d \leq 2$, X may not be transient. However, under the following assumption, X will be also transient for $d \leq 1$.

H: The potential density u of S satisfied the following: There exist constants $c > 0$ and $\gamma < d/2$ such that

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Here is our main result. For $d \geq 3$, we do not assume anything else besides our standing assumption. For $d \leq 2$, we need to assume the additional **H**.

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Let $d \geq 1$ and D a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Let G_D be the Green function of the subordinate Brownian motion X in D. Then

$$G_D(x,y) \asymp \left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}\ell(|x-y|^{-2})}{\sqrt{\ell((\delta_D(x))^{-2})\ell((\delta_D(y))^{-2})}\,|x-y|^{\alpha}}\right) \times \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}}.$$

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Examples

When $\phi(\lambda) = \lambda^{\alpha/2}$, $\ell(\lambda) = 1$;

When $\phi(\lambda) = (\lambda+1)^{\alpha/2} - 1$, $\ell(\lambda) = ((\lambda+1)^{\alpha/2} - 1)\lambda^{\alpha/2}$;

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When $\phi(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{\gamma/2}$, where $\alpha \in (0, 2)$, $\gamma \in (0, 2-\alpha]$, then $\ell(\lambda) = (\log(1+\lambda))^{\gamma/2}$;

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Outline

By spatial homogeneity we may write $G(x, y) = G(x - y)$ where the function G is given by the following formula

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} u(t) dt, \qquad x \in \mathbb{R}^d,$$

where u is the potential density of S Since u is decreasing, using this formula we see that G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

The Lévy measure of the process X has a density J, called the Lévy density, given by

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mu(t) dt, \qquad x \in \mathbb{R}^d.$$

Thus $J(x) = j(|x|)$ with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt, \qquad r > 0.$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$. We will sometimes use notation $J(x, y)$ for $J(x - y)$.

By using our standing assumption, we can apply the Tauberian theorem and the monotone density theorem to get asymptotic behaviors of u and μ at 0. Using these, one can get the following asymptotic behaviors of G and J at the origin.

Theorem

$$G(x) \sim c_1 \frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})}, \qquad |x| \to 0$$

$$j(r) \sim c_2 \frac{\ell(r^{-2})}{r^{d+\alpha}}, \qquad r \to 0$$

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Using these and some other facts, we have the following

Harnack inequality

There exist $R_1 = R_1(\alpha, \ell) \in (0, 1)$ and $c = c(\alpha, \ell) > 0$ such that for every $r \in (0, R_1)$, every $x_0 \in \mathbb{R}^d$, and every nonnegative function u on $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X we have

$$\sup_{y \in B(x_0, r/2)} u(y) \le c \inf_{y \in B(x_0, r/2)} u(y).$$

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$$\sup_{y \in B(x_0, r/2)} u(y) \le c \inf_{y \in B(x_0, r/2)} u(y).$$

An open set D in $\mathbb{R}^d$ is κ-fat with the the characteristics (R_0, κ) if each $Q \in \partial D$ and $r \in (0, R_0)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. Any $C^{1,1}$ open set is κ-fat.

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Boundary Harnack Principle

Suppose that D is a κ-fat open set with the characteristics (R_0, κ). There exist constants $R_2 = R_2(R_0, \kappa, \alpha, \ell) \le R_1$ and $c = c(R_0, \kappa, \alpha, \ell) > 1$ such that, if $r \in (0, R_2]$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have

$$c^{-1}\frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le c\,\frac{u(A_r(Q))}{v(A_r(Q))}, \qquad x \in D \cap B(Q, \frac{r}{2}).$$

Fix $z_0 \in D$ with $\kappa R_4 < \delta_D(z_0) < R_2$ and let $\varepsilon_1 := \kappa R_2/24$. For $x, y \in D$, we define $r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and $\mathcal{B}(x,y)$ to be

$$\left\{A \in D : \delta_D(A) > \frac{\kappa}{2} r(x,y),\ |x-A| \vee |y-A| < 5r(x,y)\right\}$$

when $r(x,y) < \varepsilon_1$, and to be $\{z_0\}$ when $r(x,y) \geq \varepsilon_1$.

Define

$$g(x) = G_D(x, z_0) \wedge C$$

for some appropriate constant C.

Fix $z_0 \in D$ with $\kappa R_4 < \delta_D(z_0) < R_2$ and let $\varepsilon_1 := \kappa R_2/24$. For $x, y \in D$, we define $r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and $\mathcal{B}(x,y)$ to be

$$\left\{A \in D : \delta_D(A) > \frac{\kappa}{2} r(x,y),\ |x-A| \vee |y-A| < 5r(x,y)\right\}$$

when $r(x,y) < \varepsilon_1$, and to be $\{z_0\}$ when $r(x,y) \geq \varepsilon_1$.

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$$g(x) = G_D(x, z_0) \wedge C$$

for some appropriate constant C.

Using the Green function estimates, Harnack inequality, boundary Harnack principle, and some routine (by now) arguments we get the following

Theorem

Suppose that D is a bounded κ-fat open set with the characteristics (R_0, κ). Then on $D \times D$

$$G_D(x,y) \asymp \frac{g(x)g(y)}{g(A)^2 |x-y|^{d-\alpha} \ell(|x-y|^{-2})} \quad A \in \mathcal{B}(x,y).$$

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Outline

Recall that $W = (W^1, \dots, W^d)$ is a d-dimensional Brownian motion and $S = (S_t : t \geq 0)$ an independent subordinator with the Laplace exponent ϕ satisfying our standing assumption. $X = (X_t : t \geq 0)$ is the d-dimensional subordinate Brownian motion defined by $X_t = W_{S_t}$.

Let $Z = (Z_t : t \geq 0)$ be a one-dimensional Brownian motion defined as $Z_t := W^d_{S_t}$. Then Z is a one-dimensional subordinate Brownian motion with Lévy exponent $\Phi(\theta) = \phi(|\theta|^2)$.

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Let $\overline{Z}_t := \sup\{0 \vee Z_s : 0 \leq s \leq t\}$ and let L_t be a local time of $\overline{Z} - Z$ at 0. L is also called a local time of the process Z reflected at the supremum.

Then the right continuous inverse L_t^{-1} of L is a possibly killed subordinator and is called the ladder time process of Z. The process $\overline{Z}_{L_t^{-1}}$ is also a possibly killed subordinator and is called the ladder height process of X.

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Proposition

The Laplace exponent χ of the ladder height process of the subordinate Brownian motion Z is also a complete Bernstein function.

We use V to denote the renewal function of ladder height process of Z. We will use v to denote its density. By the proposition above, v is smooth.

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Using our standing assumption and the formula

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2}d\theta\right), \quad \forall\lambda > 0,$$

one can get

$$\chi(\lambda) \sim \lambda^{\alpha/2}(\ell(\lambda^2))^{1/2}, \quad \lambda \to \infty.$$

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As $t \to 0$, we have

$$V(t) \sim c\, t^{\alpha/2} \quad \text{and} \quad v(t) \sim c\, t^{\alpha/2-1}.$$

The following result follows from a result of Silverstein:

Theorem

Let $w(x) := V((x_d)^+)$. Then w is harmonic in $\mathbb{R}^d_+$ with respect to X and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ with respect to X.

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Recall that harmonic functions are defined probabilistically. We now derive some analytic properties of w.

Define the operator $(\mathcal{A}, \mathfrak{D}(\mathcal{A}))$ as follows:

$$\mathcal{A}w(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} (w(y) - w(x)) j(|y-x|) \, dy$$

and the domain $\mathfrak{D}(\mathcal{A})$ consists of all real-valued functions w on $\mathbb{R}^d$ such that the limit above exists and is finite.

It is well known that $C_0^2 \subset \mathfrak{D}(\mathcal{A})$, and $\mathcal{A}$ restricted to C_0^2 coincides with the infinitesimal generator of the process X.

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Let $w(x) = V((x_d)^+)$. Then $\mathcal{A}w(x)$ is well defined for all $x \in \mathbb{R}^d_+$, and $\mathcal{A}w(x) = 0$ for all $x \in \mathbb{R}^d_+$.

From now on we fix a $C^{1,1}$ open set with $C^{1,1}$ characteristics (R, Λ). The following result requires more than 5 pages of calculations

Lemma

Fix $Q \in \partial D$ and let

$$h(y) := V\left(\delta_D(y)\right) \mathbf{1}_{D \cap B(Q,R)}(y).$$

There exist $C = C(\alpha, \Lambda, R, \ell) > 0$ and $R_3 \leq R/4$ independent of the point $Q \in \partial D$ such that $\mathcal{A}h$ is well defined in $D \cap B(Q, R_3)$ and

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We define for $a, b > 0$

$$D_Q(a,b) := \left\{ y \in D : a > \rho_Q(y) > 0, \, |\tilde{y}| < b \right\}.$$

Using the result above we can get the following

Lemma

There are constants $R_4 = R_4(R, \Lambda, \alpha, \ell) \in (0, R_3/(4\sqrt{1+(1+\Lambda)^2}))$ and $c_i = c_i(R, \Lambda, \alpha) > 0$, $i = 1, 2$, such that for every $r \leq R_4$, $Q \in \partial D$ and $x \in D_Q(r, r)$,

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Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Then there exists $c = c(D, \alpha) > 0$ such that for all $x \in D$,

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or equivalently (with a possibly different constant)

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Now our main result follows immediately from this and our Green function estimates for bounded κ-fat sets. In fact, our main result can be written following compact form

$$G_D(x,y) \asymp \left(1 \wedge \frac{V(\delta_D(x))V(\delta_D(y))}{V(|x-y|)^2}\right) G(x,y).$$

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