Some results on Navier-Stokes equations

Zhongmin QIAN

Mathematical Institute, The University of Oxford

19 July 2010, Beijing Normal University

Introduction

- The Navier-Stokes equations in dimension 3

$$\begin{aligned} \frac{\partial}{\partial t}u^i + \sum_{j=1}^{3} u^j \frac{\partial}{\partial x^j} u^i &= \nu \Delta u^i - \frac{\partial}{\partial x^i} p, \quad i = 1, 2, 3; \\ \sum_{j=1}^{3} \frac{\partial}{\partial x^j} u^j &= 0, \end{aligned}$$

  $v > 0$ is called the viscosity constant, $u = (u^1, u^2, u^3)$ is the velocity vector field of a fluid, p is the pressure which maintains the incompressibility.

- In vector form

$$\begin{aligned} \frac{\partial}{\partial t}u + (u \cdot \nabla)u &= \nu \Delta u - \nabla p \ ; \\ \nabla \cdot u &= 0 \ . \end{aligned}$$

- Boundary conditions have to be supplied – the common one is the non-slip condition. We however consider the simplest case (unfortunately no physical fluids), that is, $x \to u(x,t)$ is periodic, so we consider the NS on the torus $\mathbb{T}^3$ of dimension 3.
- There is a weak solution for the initial problem.
- There is a smooth solution on $\mathbb{T}^3 \times [0, T^*)$ (where $T^* > 0$ depending on the initial data) as long as the initial velocity vector field is smooth.

Geometry of the velocity vector field

The geometry determined by the velocity vector field u may be the key to understand the dynamics of the fluid, such as the global existence.

- The total derivative ∇u may be decomposed into two parts:
- The *vorticity* of u: $\omega = \nabla \times u$
- The symmetric tensor of the rate-of-strain:

$$R_i^j = \frac{1}{2}\left(\frac{\partial}{\partial x^i}u^j + \frac{\partial}{\partial x^j}u^i\right)$$

  which is the Bakry-Emery curvature corresponding to $L = \nu\Delta - u \cdot \nabla$.
- The *vorticity* of the vorticity:

$$\psi = \nabla \times \omega = -\Delta u.$$

- Describe the geometry of three vector fields (u, ω, ψ), such as the volume $\langle u, \omega \times \psi \rangle$.

Energy inequality and weak solutions

Dot the NS with $2u$ to obtain

$$\frac{\partial}{\partial t}|u|^2 + \left\langle u, \nabla |u|^2 \right\rangle = 2\nu \left\langle u, \Delta u \right\rangle - 2 \left\langle u, \nabla p \right\rangle .$$

Integrating the above equality over $\mathbb{T}^3$, and using the facts that

$$\begin{aligned} \int_{\mathbb{T}^3} \left\langle u, \nabla |u|^2 \right\rangle &= \int_{\mathbb{T}^3} |u|^2 \nabla \cdot u = 0; \\ \int_{\mathbb{T}^3} \left\langle u, \nabla p \right\rangle &= \int_{\mathbb{T}^3} p \nabla \cdot u = 0 \end{aligned}$$

(the non-linear terms) and that

$$\int_{\mathbb{T}^3} \left\langle u, \Delta u \right\rangle = -\int_{\mathbb{T}^3} \left\langle u, \nabla \times \nabla \times u \right\rangle = -\int_{\mathbb{T}^3} |\nabla \times u|^2$$

to obtain

$$\frac{\partial}{\partial t}||u||^2 = -2\nu ||\omega||^2 \tag{1}$$

called the energy balance equation, where $\omega = \nabla \times u$ the vorticity.

It follows the energy inequality

$$||u(\cdot, T)||^2 + 2\nu \int_0^T ||\nabla u(\cdot, t)||^2 dt \leq ||u_0||^2 \tag{2}$$

which takes exactly the same form as the heat equation.
Together with the usual Galerkin's approximations, one shows the existence of weak solutions.
For global strong solutions we have the following classical result:

Theorem

Let $u_0 \in C^\infty(\mathbb{T}^3)$, and $\{u(t) : t < T^*\}$ be the maximal strong solution of the Navier-Stokes equation on $\mathbb{T}^3$. If $\sup_{t<T^*} \sup_{x\in\mathbb{T}^3} |u(x, t)| < \infty$, then $T^* = \infty$, so u is a global strong solution.

The pressure and the rate-of-strain

Applying $\nabla\cdot$ to the Navier-Stokes equations we obtain

$$\begin{aligned}\Delta p &= -\nabla\cdot((u\cdot\nabla)u) = -\sum_{i,j}\frac{\partial u^j}{\partial x^i}\frac{\partial u^i}{\partial x^j} \\ &= \frac{1}{2}|\omega|^2 - |R^i_j|^2 \ . \end{aligned} \tag{3}$$

Integrating the equality over $\mathbb{T}^3$ to obtain

$$||R^i_j|| = \frac{\sqrt{2}}{2}||\omega||. \tag{4}$$

The vorticity equation

- The Bochner identity for $L = \nu\Delta - u \cdot \nabla$ takes form of the vorticity equation.
- Applying $\nabla\times$ to the Navier-Stokes equations: since $\nabla \times \nabla p = 0$

$$\frac{\partial}{\partial t}\omega + \nabla \times ((u \cdot \nabla)u) = \nu\Delta\omega.$$

- Using the vector identity

$$\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u$$

to obtain the vorticity equation

$$\frac{\partial}{\partial t}\omega + (u \cdot \nabla)\omega = \nu\Delta\omega + (\omega \cdot \nabla)u.$$

- Since

$$(\omega \cdot \nabla)u^i = \sum_j R^i_j \omega^j$$

so that

$$\frac{\partial}{\partial t}\omega + (u \cdot \nabla)\omega = \nu\Delta\omega + \sum_j R^i_j \omega^j \,. \tag{5}$$

Evolution for the Laplacian

Applying $\nabla\times$ to the vorticity equation (or equivalently applying Δ to the Navier-Stokes equation) to obtain the evolution equation for $\psi = -\Delta u = \nabla \times \omega$:

$$\begin{aligned}\frac{\partial}{\partial t}\psi + u \cdot \nabla \psi &= \nu \Delta \psi - R(\psi) - \frac{1}{2}\omega \times \psi \\ &\quad + 2\left\langle R, \nabla^2 u \right\rangle + \nabla\left(\frac{1}{2}|\omega|^2 - |R^i_j|^2\right).\end{aligned}$$

It follows from this evolution equation we can deduce the following

Theorem

Let $\{u(t) : t < T^*\}$ be the maximal strong solution of the Navier-Stokes equation on $\mathbb{T}^3$. Suppose that $\sup_{t<T^*} ||\omega(t)|| < \infty$, then $T^* = \infty$. That is, the maximal solution is a global solution.

Ricci flow for the NS equations

We can also devise the evolution equation for the symmetric tensor of the rate-of-strain:

$$L_t R_i^j = -R_i^k R_k^j + \frac{1}{4} \sum_{a,b,k \leq 3} \delta_{aik} \delta_{bjk} \omega^a \omega^b - \nabla_i \nabla^j p \tag{6}$$

where

$$L_t = \frac{\partial}{\partial t} - \nu \Delta + u(t) \cdot \nabla$$

is the heat operator associated with $L = \nu \Delta - u \cdot \nabla$.

Estimating the vorticity

Let $u = \{u(\cdot, t) : t < T^*\}$ be the maximal solution of the Navier-Stokes equation on $\mathbb{T}^3$.

Lemma

Let $q \geq 1$ and $\varepsilon > 0$ be two constants, and $F_\varepsilon = \left(|\omega|^2 + \varepsilon\right)^{q/2}$. Then

$$\frac{d}{dt}\int_{\mathbb{T}^3} F_\varepsilon \leq -\frac{4\nu\,(q-1)}{q}\int_{\mathbb{T}^3} |\nabla\sqrt{F_\varepsilon}|^2 + q\int_{\mathbb{T}^3} \frac{F_\varepsilon}{|\omega|^2+\varepsilon}\langle\omega, R\omega\rangle \qquad (7)$$

as long as $t < T^*$.

Therefore

Corollary

For any $q \geq 1$

$$\frac{d}{dt}||\omega||_q^q \leq -\frac{4(q-1)}{q}\nu\int_{\mathbb{T}^3} |\nabla|\omega|^{q/2}|^2 + q\int_{\mathbb{T}^3} |\omega|^{q-1}|R\omega|. \qquad (8)$$

Monotonicity of the vorticity

We can establish the following

Theorem

Let $u(t)$ ($t < T^*$) be the strong solution of the Navier-Stokes equations on $\mathbb{T}^3$. Then

$$t \to ||\omega(t)||_{L^1} + \frac{\sqrt{2}}{4\nu}||u(t)||^2 \tag{9}$$

is decreasing on $[0, T^*)$.

Proof of Theorem. Apply (8) with $q = 1$ we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}^3}|\omega| &\le \int_{\mathbb{T}^3}|R(\omega)| \le ||\omega||||R^i_j|| \\ &= \frac{1}{\sqrt{2}}||\omega||^2. \end{aligned}$$

Now, we use the energy balance equation

$$\frac{d}{dt}||u||^2 = -2\nu||\omega||^2$$

and replace $||\omega||^2$ by $-\frac{1}{2\nu}\frac{d}{dt}||u||^2$, we thus obtain

$$\frac{d}{dt}||\omega||_1 \le -\frac{\sqrt{2}}{4\nu}\frac{d}{dt}||u||^2$$

which proves the theorem.

Global existence

Similarly we have

Theorem

Let u be a strong solution of the Navier-Stokes equation on $\mathbb{T}^3$. Then there are two constants C_1, C_2 depending only on $q > 1$, such that

$$||\omega(t)||_q^q \leq e^{C_1 t}||\omega_0||_q^q + C_2 \int_0^t e^{C_1(t-s)}||\omega(s)||^{2(2q-1)} ds \tag{10}$$

for all $t < T^*$.

Therefore

Corollary

Let $T \leq T^*$ or $T < \infty$ if $T^* = \infty$. Then there is a constant K depending only on $q \geq 1$ and T, such that

$$\sup_{t \in [0,T)} ||\omega(t)||_q \leq K \left(||\omega_0||_q + \int_0^T ||\omega(s)||^{2(2q-1)} ds \right) . \tag{11}$$

As a consequence

Theorem

Let $u_0 \in C^\infty(\mathbb{T}^3)$ and $\{u(t) : t < T^*\}$ be the maximum strong solution of the Navier-Stokes equation with initial u_0. If $\int_0^{T^*} ||\omega(s)||^6 ds < \infty$, then $T^* = \infty$.

Proof.

Suppose $T^* < \infty$. Then by applying (11) to $q = 2$, we can see that $\sup_{t\in[0,T^*)} ||\omega(t)|| < \infty$. □