Almost Sure Exponential Stability of Numerical Methods for Hybrid SDEs

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Outline

1. Hybrid SDEs
   - Stability of true solutions
   - Stability of the EM method
   - Examples
2. New Problem
   - Stable hybrid SDE without linear growth condition
   - The EM method blows up
3. The Backward Euler–Maruyama Method
   - Definition
   - Stability of the BEM

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The issue that we address in this talk is:

Can a numerical method reproduce the stability behaviour of the underlying hybrid SDE?

Brief history

- Saito and Mitsui (SIAM J. Numer. Anal. **33**, 1996): linear SDE and mean-square exponential stability.
- Many works on mean-square stability of numerical methods.
- Higham (SIAM J. Numer. Anal. **38**, 2000): linear SDE, a.s. exp. stability, but a revised EM method.
- Higham, Mao and Yuan (SIAM Journal on Numerical Analysis **45**, 2007): linear SDE and a.s. exp. stability.
- Pang, Deng and Mao (J. Computational and Applied Math. **213**, 2008): nonlinear hybrid SDE, a.s. exp. stability.

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Consider an n-dimensional nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t) \qquad (1.1)$$

on $t \geq 0$ with initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$, where

$$f : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n.$$

Assume that f and g are smooth enough so that the hybrid SDE (1.1) has a unique global solution $x(t)$ on $[0, \infty)$.

Theorem

Assume that for each $i \in \mathbb{S}$,

$$\lambda_i := \sup_{x\in\mathbb{R}^n, x\neq 0} \left(\frac{\langle x, f(x,i)\rangle + \frac{1}{2}|g(x,i)|^2}{|x|^2} - \frac{\langle x, g(x,i)\rangle^2}{|x|^4} \right) < \infty. \qquad (1.2)$$

Then the solution of the hybrid SDE (1.1) obeys

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \le \sum_{i\in\mathbb{S}} \pi_i \lambda_i \quad a.s. \qquad (1.3)$$

In particular, if $\sum_{i\in\mathbb{S}} \pi_i \lambda_i < 0$, then the hybrid SDE (1.1) is almost surely exponentially stable.

Linear hybrid SDEs

$$dx(t) = A(r(t))x(t)dt + G(r(t))x(t)dB(t). \tag{1.4}$$

Here $A, G : \mathbb{S} \to \mathbb{R}^{n \times n}$. For convenience, we will write $A(i) = A_i$ and $G(i) = G_i$. This SDE corresponds to

$$f(x,i) = A_i x \quad \text{and} \quad g(x,i) = G_i x \quad (x,i) \in \mathbb{R}^n \times \mathbb{S},$$

in (1.1).

Corollary

Assume that every $G_i + G_i^T$ ($i \in \mathbb{S}$) is either non-positive definite or non-negative definite and define

$$\lambda_i := \frac{1}{2}\lambda_{\max}(A_i + A_i^T) + \frac{1}{2}\|G_i\|^2 - \frac{1}{4}\Big[|\lambda_{\max}(G_i + G_i^T)| \wedge |\lambda_{\min}(G_i + G_i^T)|\Big]^2 .$$

If

$$-\lambda := \sum_{i\in\mathbb{S}} \pi_i \lambda_i < 0, \tag{1.5}$$

then the linear hybrid SDE (1.4) is almost surely exponentially stable.

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Given a step size $\Delta > 0$, the Euler-Maruyama method is to compute the discrete approximations $X_k \approx x(k\Delta)$ by setting $X_0 = x_0$ and forming

$$X_{k+1} = X_k + f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k, \tag{1.6}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$.

Linear growth condition

For the EM method to reproduce the stability property, we need the linear growth condition:

$$|f(x,i)| \vee |g(x,i)| \leq K|x|, \quad \forall (x,i) \in \mathbb{R}^n \times \mathbb{S}. \tag{1.7}$$

Theorem

Let (1.7) and (1.2) hold and assume that

$$-\lambda := \sum_{i\in\mathbb{S}} \pi_i \lambda_i < 0. \tag{1.8}$$

Then for any $\varepsilon \in (0,\lambda)$ *there is a constant* $\Delta^\star \in (0,1)$ *such that for any* $0 < \Delta < \Delta^\star$ *the EM approximation (1.6) satisfies*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log(|X_k|) \le -(\lambda - \varepsilon) \quad \text{a.s.} \tag{1.9}$$

The EM method applying to the linear hybrid SDE (1.4) forms approximation $X_k \approx x(t_k)$, with $t_k = k\Delta$, by setting $X_0 = x_0,\ r_0^\Delta = i_0$ and, generally,

$$X_{k+1} = X_k + A(r_k^\Delta)X_k\Delta + G(r_k^\Delta)X_k\Delta B_k. \tag{1.10}$$

Corollary

Assume that every $G_i + G_i^T$ $(i \in \mathbb{S})$ is either non-positive definite or non-negative definite and define

$$\begin{aligned}\lambda_i \;&:=\; \tfrac{1}{2}\lambda_{\max}(A_i + A_i^T) + \tfrac{1}{2}\|G_i\|^2 \\ &-\; \tfrac{1}{4}\Big[|\lambda_{\max}(G_i + G_i^T)| \wedge |\lambda_{\min}(G_i + G_i^T)|\Big]^2.\end{aligned}$$

If $\sum_{i\in\mathbb{S}} \pi_i\lambda_i < 0$, then for any $\varepsilon \in (0,)$ there is a constant $\Delta^\star \in (0,1)$ such that for any $0 < \Delta < \Delta^\star$ the EM approximation (1.10) is almost surely exponentially stable.

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In the following examples, we let $r(t)$ be a Markov chain with $\mathbb{S} = \{1, 2\}$ and

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

Its stationary distribution is $\pi = (4/5, 1/5)$.

Consider a scalar nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t),$$

where

$$f(x,1) = \sin x, \quad g(x,1) = 2x, \quad f(x,2) = x\cos x, \quad g(x,2) = x.$$

To apply the stability theorem we compute

$$\lambda_1 = \sup_{x \in \mathbb{R}, x \neq 0} \left(\frac{\langle x, f(x,1)\rangle + \frac{1}{2}|g(x,1)|^2}{|x|^2} - \frac{\langle x, g(x,i)\rangle^2}{|x|^4} \right) = -1,$$

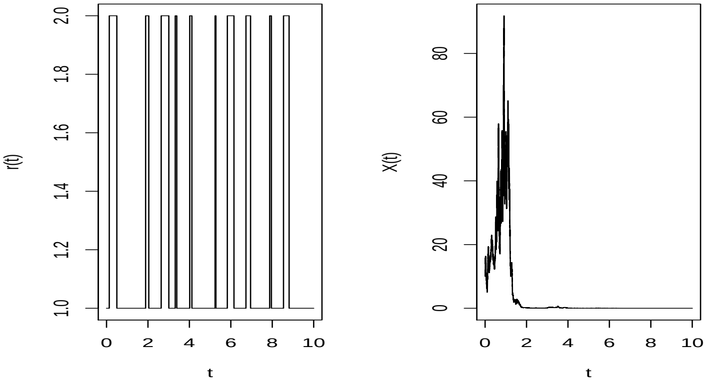
and

$$\lambda_2 = 0.5$$

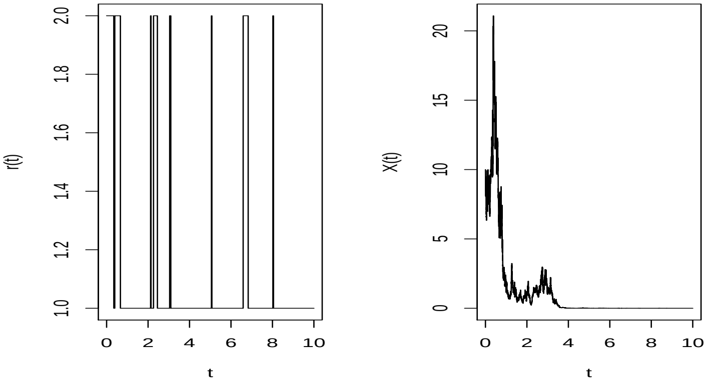
Consequently

$$\pi_1\lambda_1 + \pi_2\lambda_2 = -0.7.$$

Hence the nonlinear hybrid SDE is almost surely exponentially stable and so is the EM method provided the stepsize is sufficiently small.



$$\Delta = 0.001, \ x(0) = 10, \ r(0) = 1$$



$$\Delta = 0.001, \; x(0) = 10, \; r(0) = 2$$

Consider a 2-dimensional linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + G(r(t))x(t)dB(t),$$

where

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -0.5 & 0.5 \\ 0 & -0.5 \end{pmatrix}.$$

To apply the Corollary, we observe that

$$G_1 + G_1^T = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad G_2 + G_2^T = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix}$$

are positive and negative definite, respectively. Compute

$$\begin{aligned} \lambda_1 &:= \tfrac{1}{2}\lambda_{\max}(A_1 + A_1^T) + \tfrac{1}{2}\|G_1\|^2 \\ &- \tfrac{1}{4}\left[|\lambda_{\max}(G_1 + G_1^T)| \wedge |\lambda_{\min}(G_1 + G_1^T)|\right]^2 \\ &= -1.5 \end{aligned}$$

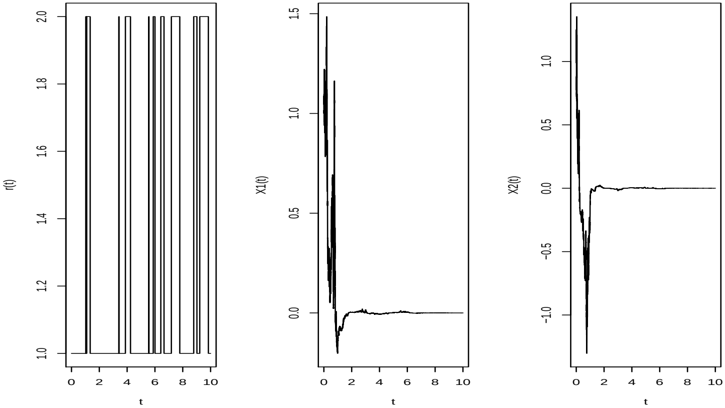
and

$$\lambda_2 = 0.3273$$

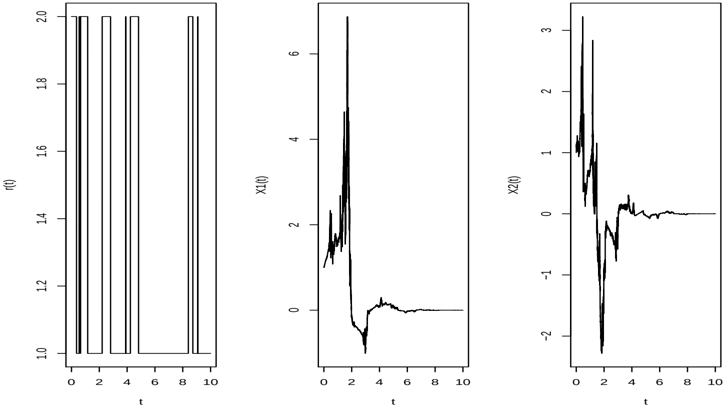
Consequently

$$\pi_1\lambda_1 + \pi_2\lambda_2 = -1.13454$$

Hence the 2-dimensional linear hybrid SDE is almost surely exponentially stable and so is the EM method provided the stepsize is sufficiently small.



$$\Delta = 0.001, \; x_1(0) = 1, \; x_2(0) = 1, \; r(0) = 1$$



$$\Delta = 0.001,\ x_1(0) = 1,\ x_2(0) = 1,\ r(0) = 2$$

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Consider the scalar hybrid cubic SDE

$$dx(t) = [\alpha(r(t))x(t) - x^3(t)]dt + \beta(r(t))x(t)dB(t), \qquad (2.1)$$

where

$$\alpha(1) = 1, \quad \beta(1) = 2, \quad \alpha(2) = 0.5, \quad \beta(2) = 1,$$

and $r(t)$ is a Markov chain with the state space $\mathbb{S} = \{1, 2\}$ and the generator

$$\Gamma = \begin{bmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{bmatrix},$$

where $\gamma_{12} > 0$ and $\gamma_{21} > 0$.

It is easy to see that the unique stationary distribution $\pi = (\pi_1, \pi_2) \in \mathbb{R}^{1\times 2}$ of the Markov chain is given by

$$\pi_1 = \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}}, \quad \pi_2 = \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}}.$$

It follows from the stability theorem above that

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \log |x(t)| &\le \pi_1[\alpha(1) - 0.5\beta^2(1)] + \pi_2[\alpha(2) - 0.5\beta^2(2)] \\ &= -\frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}} \quad \text{a.s.} \end{aligned} \tag{2.2}$$

In other words, the SDE (2.1) is almost surely exponentially stable.

If the EM method is applied to this hybrid cubic SDE, will it recover the stability property?

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Applying the EM to the hybrid cubic SDE produces

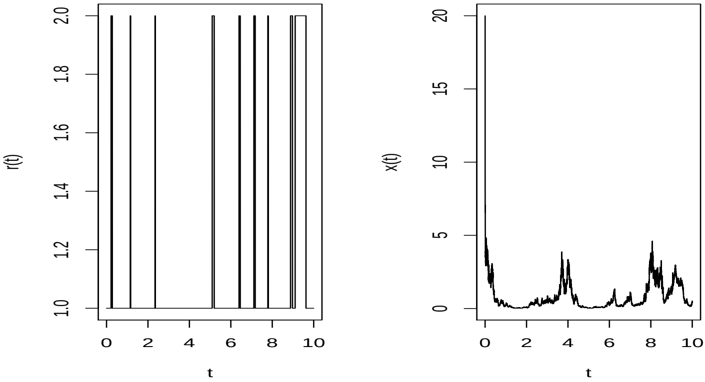
$$X_{k+1} = X_k \left(1 + \Delta\alpha(r_k^\Delta) - \Delta X_k^2 + \beta(r_k^\Delta)\Delta B_k \right). \tag{2.3}$$

Lemma

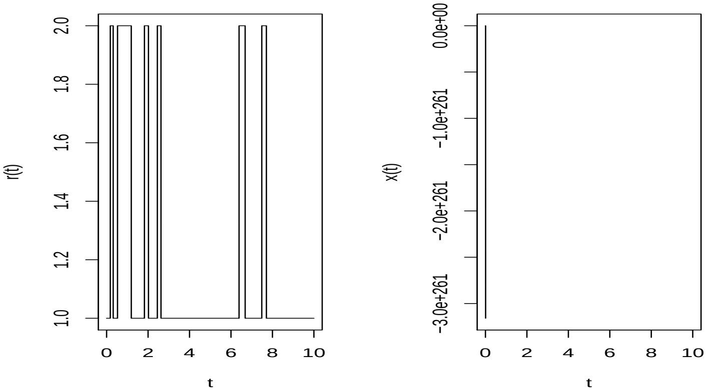
Given any initial value $x(0) \neq 0$ and any $\Delta > 0$,

$$\mathbb{P}\left(\lim_{k\to\infty} |X_k| = \infty \right) > 0.$$

The following two figures show the results of two computer simulations based on the EM method with step size $\Delta = 0.001$ and initial values $(x(0), r(0)) = (20, 1)$ and $(50, 1)$, respectively. Both simulations show that the EM method does not capture the stability property of the underlying SDE (2.1), while the second simulation shows that the EM method can blow up very quickly.



$$\Delta = 0.001,\ x(0) = 20,\ r(0) = 1,\ \gamma_{12} = 1,\ \gamma_{21} = 4.$$



$$\Delta = 0.001, \ x(0) = 50, \ r(0) = 1, \ \gamma_{12} = 1, \ \gamma_{21} = 4.$$

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Definition of the BEM

Given a step size $\Delta > 0$, the The Backward Euler–Maruyama (BEM) method produces approximations $X_k \approx x(t_k)$, where $X_0 = x_0,\ r_0^\Delta = i_0$ and, generally,

$$X_{k+1} = X_k + f(X_{k+1}, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k, \quad k \geq 0. \tag{3.1}$$

The BEM method is implicit as for every step given X_k, equation (3.1) needs to be solved for X_{k+1}. For this purpose, some conditions need to be imposed on f.

The one-side Lipschitz condition

There is a constant $\mu \in \mathbb{R}$ such that

$$\langle x-y, f(x,i)-f(y,i)\rangle \le \mu|x-y|^2, \quad \forall x,y \in \mathbb{R}^n,\ i \in \mathbb{S}. \quad (3.2)$$

Under this condition, it is known that equation (3.1) can be solved uniquely for X_{k+1} given X_k as long as the step size $\Delta < 1/(1+2|\mu|)$.

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Assumption

Assume that

$$|f(0,i)| = 0, \quad \forall i \in \mathbb{S}.$$

and, moreover, there is an $h > 0$ such that

$$|g(x,i)| \le h|x| \quad \forall (x,i) \in \mathbb{R}^n \times \mathbb{S}.$$

Theorem

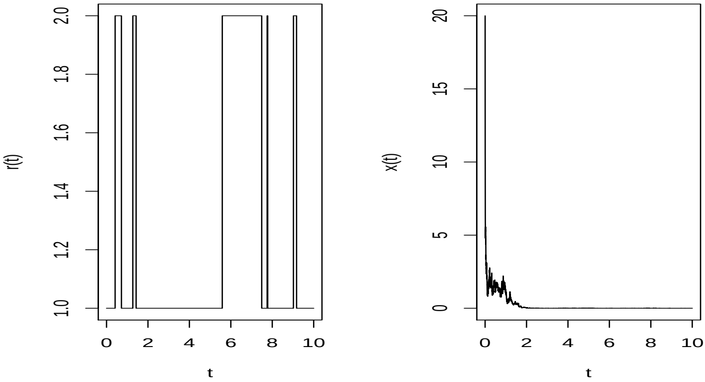
Let the Assumption above hold. Let also the one-side Lipschtiz condition (3.2) and and condition (1.2) hold. Assume that

$$-\lambda := \sum_{i\in\mathbb{S}} \pi_i \lambda_i < 0.$$

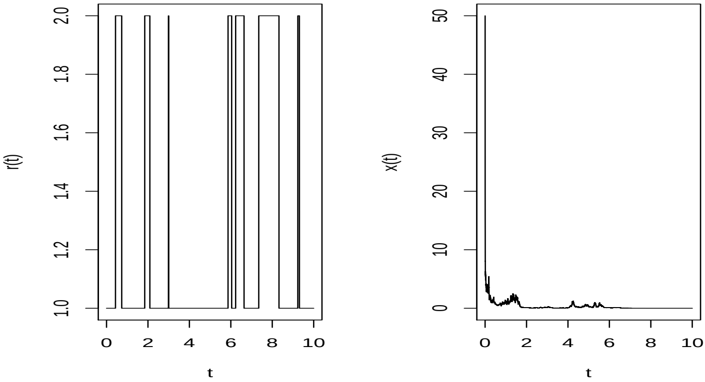
Then for any $\varepsilon \in (0, \lambda)$, *there is a* $\Delta^\star > 0$ *such that for any* $\Delta < \Delta^\star$, *the BEM method (3.1) has the property that*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log |X_k| \le -\lambda + \varepsilon < 0 \quad a.s. \tag{3.3}$$

Let us now return to the hybrid cubic SDE (2.1). Our theory shows that the BEM method can reproduce the stability. The two simulations shown in the following figures are based on the BEM method with initial values set as before to $(x(0), r(0)) = (20, 1)$ and $(50, 1)$, respectively. Both figures show clearly that the BEM method reproduces the almost sure exponential stability of the underlying SDE (2.1).



$$\Delta = 0.001, \; x(0) = 20, \; r(0) = 1, \; \gamma_{12} = 1, \; \gamma_{21} = 4.$$



$$\Delta = 0.001, \; x(0) = 50, \; r(0) = 1, \; \gamma_{12} = 1, \; \gamma_{21} = 4.$$