Transportation-information inequalities for continuum Gibbs measures

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   - Gibbs measure and generator of the Glauber dynamic
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   - Lipschitzian norm of $(-\mathcal{L}_\Lambda^\eta)^{-1}$
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4. W_1I for discrete spin system
   - Discrete spin system
   - W_1I for discrete spin system

L^p-Wasserstein distance

$(\mathcal{X}, \mathcal{B})$ a Polish space, d a lower semi-continuous metric on $\mathcal{X} \times \mathcal{X}$.

$\mathcal{M}_1(\mathcal{X})$ the space of all probability measures on $\mathcal{X}$.

$\mathcal{M}_1^p(\mathcal{X}, d) := \{\nu \in \mathcal{M}_1(\mathcal{X});\ \int d^p(x, x_0) d\nu < +\infty\}$, where x_0 is some fixed point of $\mathcal{X}$.

Given $p \geq 1$ and two probability measures μ and ν on $\mathcal{X}$, we define the quantity

$$W_{p,d}(\mu, \nu) = \inf \left(\iint d(x, y)^p d\pi(x, y) \right)^{1/p}.$$

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Fisher-Donsker-Varadhan information

Given a Dirichlet form $\mathcal{E}$ on $L^2(\mu) := L^2(\mathcal{X}, \mu)$ with domain $\mathcal{D}(\mathcal{E})$, the Fisher-Donsker-Varadhan information of ν with respect to μ is given as:

$$I(\nu|\mu) = \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f\mu, \sqrt{f} \in \mathcal{D}(\mathcal{E}) \\ +\infty & \text{otherwise} \end{cases}$$

Transportation-information inequality

α a nondecreasing left-continuous function on $\mathbb{R}^+ = [0, +\infty)$ which vanishes at 0.

Probability measure μ satisfies a transportation-information inequality $W_1 I$ if

$$\alpha(W_{1,d}(\nu, \mu)) \leq I(\nu|\mu), \ \forall \nu \in M_1(\mathcal{X}),$$

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GLWY's equivalence for $W_1 I$-inequality

The following properties are equivalent:

(a) The transportation-information inequality holds.

(b) The following concentration inequality holds for each $g \in b\mathcal{B}$ with $\|g\|_{\mathrm{Lip}(d)} \le 1$ and any initial distribution $\nu \ll \mu$,

$$\mathbb{P}_\nu\left(\frac{1}{t}\int_0^t g(X_s)ds > \mu(g) + r\right) \le \|\frac{d\nu}{d\mu}\|_2 e^{-t\alpha(r)}, \ \forall\, t, r > 0,$$

where $\|\cdot\|_2$ is the norm of $L^2(\mu)$.

Configuration space

Ω the space of all point measures $\sum_i \delta_{x_i}$(finite or countable) with x_i different in $\mathbb{R}^d$;

$\mathcal{F}_A = \sigma\Big(\omega(B) : B(\text{Borelian}) \subset A\Big)$ for each $A \in \mathcal{B}_b(\mathbb{R}^d)$

Borel σ field on $\mathbb{R}^d$ is $\mathcal{F} = \mathcal{F}_{\mathbb{R}^d}$

Given a bounded open subset Λ of $\mathbb{R}^d$ and $\omega \in \Omega$,

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Poisson point measures

Give the activity $z > 0$, let P be the law of Poisson point process on $\mathbb{R}^d$ with intensity measure zdx.

The image measure P_Λ of P by $\omega \to \omega_\Lambda$ is the law of Poisson point process on Λ with intensity measure zdx.

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The Gibbs measure

The Gibbs measure for a given boundary condition $\eta \in \Omega$ on Λ^c is a probability given by

$$\mu_\Lambda^\eta(d\omega_\Lambda) := (Z_\Lambda^\eta)^{-1} \exp\left\{-\beta H_\Lambda^\eta(\omega_\Lambda)\right\} P_\Lambda(d\omega_\Lambda)$$

where Z_Λ^η is the normalization constant, and

$$H_\Lambda^\eta(\omega_\Lambda) := \frac{1}{2}\iint_{\Lambda^2} \phi(x-y)\omega_\Lambda(dx)\omega_\Lambda(dy) + \int_\Lambda \omega_\Lambda(dx)\int_{\Lambda^c} \phi(x-y)\eta(dy)$$

is the Hamiltonian.

Difference operator

For a real $\mathcal{F}-$measurable function f, consider the difference operators

$$D_x^+ f(\omega) := f(\omega + \delta_x) - f(\omega), \quad \omega \in \Omega_\Lambda, x \in \Lambda;$$
$$D_x^- f(\omega) := f(\omega - \delta_x) - f(\omega), \quad \omega \in \Omega_\Lambda, x \in \operatorname{supp} \omega.$$

Generator and Dirichlet form

Generator:

$$\mathcal{L}_\Lambda^\eta f(\omega_\Lambda) = \int_\Lambda D_x^- f(\omega_\Lambda)\omega_\Lambda(dx) + z\int_\Lambda e^{-\beta D_x^+ H_\Lambda^\eta(\omega_\Lambda)} D_x^+ f(\omega_\Lambda) dx.$$

Dirichlet form:

$$\begin{aligned}\mathcal{E}_\Lambda^\eta(f,g) &:= \langle f, -\mathcal{L}_\Lambda^\eta g\rangle_{\mu_\Lambda^\eta} \\ &= \int_{\Omega_\Lambda} d\mu_\Lambda^\eta(\omega_\Lambda)\int_\Lambda D_x^- f(\omega_\Lambda) D_x^- g(\omega_\Lambda)\omega_\Lambda(dx) \\ &= \int_{\Omega_\Lambda} d\mu_\Lambda^\eta(\omega_\Lambda)\int_\Lambda e^{-\beta D_x^+ H_\Lambda^\eta(\omega_\Lambda)} D_x^+ f(\omega_\Lambda) D_x^+ g(\omega_\Lambda) z dx.\end{aligned}$$

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$M/M/\infty$ queue system

Generator: $\mathcal{L}f(n) = \lambda(f_{n+1} - f_n) + n(f_{n-1} - f_n)$

Invariant measure: $\mu_n = e^{-\lambda}\lambda^n/n!$

Dirichlet form: $\mathcal{E}(f,g) = \sum_{n=0}^{\infty} \lambda\mu_n(f_{n+1} - f_n)(g_{n+1} - g_n)$.

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Optimal $W_1 I$-inequality for $M/M/\infty$

Theorem. Consider ρ the classical Euclidean distance on $\mathbb{N}$. Then

$$W_{1,\rho}(\nu, \mu) \leq I + 2\sqrt{\lambda I}, \quad \forall \nu \in \mathcal{M}_1(\mathbb{N}_+),$$

where $I = I(\nu/\mu)$.

The inequality is optimal since

- Gao, Guilllin and Wu proved that $\nu(g_0) - \mu(g_0) \leq 2\sqrt{\lambda I} + I$ for $g_0(n) = n - \lambda$ is optimal (motivation).
- Take ν a Poisson distribution with parameter $a\lambda$, $a > 1$. Then $W_{1,\rho}(\nu,\mu) = 2\sqrt{\lambda I} + I = \lambda(\sqrt{a} - 1)^2$.

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Some key points

- Lipschitzian spectral gap: $||(-\mathcal{L})^{-1}||_{\mathrm{Lip}(\rho)} = 1$
- Lyapunov test function: take $V_n = \kappa^n\ (\kappa > 1)$, s.t.

$$(1+\delta)n + (1+\frac{1}{\delta})\lambda \le -a\frac{\mathcal{L}V}{V}(n) + b$$

with $a = (1+\delta)\kappa/(\kappa-1)$ and $b = ((1+\delta)\kappa + (1+\frac{1}{\delta}))\lambda$

+ for any function $V \ge 1$, if $-\frac{\mathcal{L}V}{V}$ is lower bounded, then

$$\int -\frac{\mathcal{L}V}{V}d\nu \le I(\nu|\mu),\ \forall \nu.$$

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Sketch of the Proof

Given any function g on $\mathbb{N}$ with $\mu(g) = 0$, $\|g\|_{\mathrm{Lip}(\rho)} = 1$, G satisfies $-\mathcal{L}G = g$ with $\mu(G) = 0$. For any $\delta > 0$, we have

$$\nu(g) - \mu(g) =< g, f >_\mu = \mathcal{E}(G, f) = \sum_{n=0}^{\infty} \lambda\mu_n(G_{n+1} - G_n)(f_{n+1} - f_n)$$

$$\leq \sqrt{\sum_{n=0}^{\infty} \lambda\mu_n(\sqrt{f_{n+1}} - \sqrt{f_n})^2} \cdot \sqrt{\sum_{n=0}^{\infty} \lambda\mu_n(G_{n+1} - G_n)^2(\sqrt{f_{n+1}} + \sqrt{f_n})^2}$$

$$\leq \sqrt{I\sum_{n=0}^{\infty} \lambda\mu_n\Big((1+\delta)f_{n+1} + (1+\frac{1}{\delta})f_n\Big)} = \sqrt{I\sum_{n=0}^{\infty} \mu_n f_n\Big((1+\delta)n + (1+\frac{1}{\delta})\lambda\Big)}$$

$$\leq \sqrt{I\sum_{n=0}^{\infty} \mu_n f_n\Big(-a\frac{\mathcal{L}V}{V}(n) + b\Big)} \leq \sqrt{I(aI + b)}.$$

Lipschitzian space

The metric d on Ω_Λ : for any $\omega, \omega' \in \Omega_\Lambda$,

$$d(\omega, \omega') = \|\omega - \omega'\|_{\mathrm{TV}}.$$

Given any functional $F \in r\mathcal{F}_\Lambda$, F is Lipschitzian with respect to d if

$$\|F\|_{\mathrm{Lip}(d)} := \sup_{\omega \neq \omega'} \frac{|F(\omega) - F(\omega')|}{d(\omega, \omega')} < \infty$$

$$\Longleftrightarrow \|F\|_{\mathrm{Lip}(d)} = \sup_{x \in \Lambda, \omega_\Lambda \in \Omega_\Lambda} |D_x^+ F(\omega_\Lambda)| < \infty.$$

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Upper bound of $\|(-\mathcal{L}_\Lambda^\eta)^{-1}\|_{\mathrm{Lip}(d)}$

Lemma. Suppose that the Dobrushin's uniqueness condition holds, i.e.,

$$D = z \int_{\mathbb{R}^d} (1 - e^{-\beta\varphi(x)}) dx < 1.$$

We have

$$\|(-\mathcal{L}_\Lambda^\eta)^{-1}\|_{\mathrm{Lip}(d)} \le \frac{1}{1-D}.$$

Lyapunov test function

Lemma. Take $V(\omega_\Lambda) = \kappa^{N_\Lambda(\omega_\Lambda)}$. Given any $\delta > 0$, then

$$(1+\delta)N_\Lambda(\omega_\Lambda) + (1+\frac{1}{\delta})z|\Lambda| \leq -a\frac{\mathcal{L}_\Lambda^\eta V(\omega_\Lambda)}{V(\omega_\Lambda)} + b, \quad \omega_\Lambda \in \Omega_\Lambda$$

where $a = (1+\delta)\frac{\kappa}{\kappa-1}$, $b = \big((1+\delta)\kappa + (1+\frac{1}{\delta})\big)z|\Lambda|$.

$W_1 I$-inequality for μ_Λ^η

Theorem. Suppose that the Dobrushin uniqueness condition holds, i.e.

$$D = z \int_{\mathbb{R}^d} (1 - e^{-\beta\varphi(x)})dx < 1.$$

The Gibbs measure μ_Λ^η satisfies the transportation-information inequality $W_1 I$

$$W_{1,d}(\nu, \mu_\Lambda^\eta) \leq \frac{1}{1-D}\left(I + 2\sqrt{z|\Lambda|I}\right),$$

where $I = I(\nu/\mu_\Lambda^\eta)$.

Remark. The transportation-information inequality for μ_Λ^η is sharp. If $\varphi = 0$, then $D = 0$ and $N_\Lambda(X_t)$ is just the $M/M/\infty$ queue system with $\lambda = z|\Lambda|$.

The discrete spin system

T a finite subset of $\mathbb{Z}^d$

$\gamma : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$ a nonnegative interaction function satisfying $\gamma_{ij} = \gamma_{ji}$ and $\gamma_{ii} = 0$ for all $i, j \in \mathbb{Z}^d$.

The Gibbs measure on $\mathbb{N}^T$ with boundary condition $(x_k)_{k \in T^c}$ is defined by

$$\mu_T(dx_T|x) = \frac{e^{-\frac{1}{2}\sum_{\{i,j\}\cap T\neq\emptyset} \gamma_{ij} x_i x_j}}{Z(x_{T^c})} \Pi_{i\in T} \sigma_{\lambda_i}(dx_i)$$

where $\{\sigma_{\lambda_i}(\cdot)\}_{i\in\mathbb{Z}^d}$ are the given Poisson measures on $\mathbb{N}$ with means $\{\lambda_i > 0\}_{i\in\mathbb{Z}^d}$, and $Z(x_{T^c})$ is the normalization factor.

When $T = \{i\}$, $\mu_T(dx_T|x)$ noted as $\mu_i(dx_i|\, x)$, is the Poisson distribution with parameter $\lambda_i e^{-\sum_{j\neq i} \gamma_{ij} x_j}$.

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Dobrushin interdependence matrix

$C := (c_{ij})_{i,j\in T}$ w.r.t. d_{l^1} on $\mathbb{N}$ is

$$c_{ij} = \sup_{x=x' \text{off} j} \frac{W_{1,\rho}\left(\mu_i(dx_i|x), \mu_i(dx_i'|x')\right)}{|x_j - x_j'|} = \lambda_i(1 - e^{-\gamma_{ij}}).$$

Dobrushin's uniqueness condition

$$D := \sup_{j\in T} \sum_{i\in T} c_{ij} = \sup_{j\in T} \sum_{i\in T} \lambda_i(1 - e^{-\gamma_{ij}}) < 1.$$

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The Dirichlet form of $\mathcal{E}_T$ is defined as

$$\mathcal{E}_T(g,g) := \int_{\mathbb{N}^T} \sum_{i\in T} \mathcal{E}_i(g_i,g_i)d\mu_T, \quad g\in\mathcal{D}(\mathcal{E}_T) \qquad \text{with}$$

$$\mathcal{D}(\mathcal{E}_T) := \left\{ g\in L^2(\mu_T) :\ g_i\in\mathcal{D}(\mathcal{E}_i), \mu_T - \text{a.e. } \hat{x}_i, \int_{\mathbb{N}^T}\sum_{i\in T}\mathcal{E}_i(g_i,g_i)d\mu_T < +\infty \right\}$$

where $g_i(x_i) := g(x_i, \hat{x}_i)$ with $\hat{x}_i := x_{T\setminus\{i\}}$ fixed.

Tensorization result for W_1

Lemma. Assume the Dobrushin's uniqueness condition

$$D = \sup_{j \in T} \sum_{i \in T} \lambda_i (1 - e^{-\gamma_{ij}}) < 1.$$

Then for all $\nu_T \in \mathcal{M}_1^1(\mathbb{N}^T)$,

$$W_{1,d_{l^1}}(\nu_T, \mu_T) \leq \frac{1}{1-D} \mathbb{E}^{\nu_T} \sum_{i \in T} W_{1,\rho}(\nu_i, \mu_i)$$

where ν_i is the conditional distribution of x_i knowing $(x_j)_{j \neq i}$.

Additivity property of the Fisher information

Let ν_T, μ_T be probability measures on $\mathbb{N}^T$ such that $I_T(\nu_T|\mu_T) < +\infty$, and let μ_i, ν_i be the conditional distributions of x_i knowing $\hat{x}_i$ under μ, ν respectively. Then

$$I_T(\nu_T|\mu_T) = \mathbb{E}^{\nu_T} \sum_{i \in T} I_i(\nu_i|\mu_i)$$

where $I_i(\nu_i|\mu_i)$ is the Fisher-Donsker-Varadhan information related to the Dirichlet form $(\mathcal{E}_i, \mathcal{D}(\mathcal{E}_i))$.

W_1 I-inequality for discrete spin system

Theorem. Assume the Dobrushin uniqueness condition

$$D = \sup_{j \in T} \sum_{i \in T} \lambda_i (1 - e^{-\gamma_{ij}}) < 1.$$

Then for any $\nu_T \in \mathcal{M}_1^1(\mathbb{N}^T, d_{l^1})$, it holds that

$$W_{1,d_{l^1}}(\nu_T, \mu_T) \leq \frac{1}{1-D} \left(2\sqrt{\sum_{i \in T} \lambda_i I} + I \right)$$

where $I = I_T(\nu_T | \mu_T)$.

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Thank you for your attention