Stochastic differential equations with Sobolev coefficients

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July 19–23, 2010

Outline

Let $A_0, A_1, \cdots, A_m$ be Borel measurable vector fields on $\mathbb{R}^d$, and $w_t = (w_t^1, \cdots, w_t^m)^*$ an m-dimensional standard Brownian motion. Consider the Itô SDE

$$\mathrm{d}X_t = \sum_{i=1}^{m} A_i(X_t)\,\mathrm{d}w_t^i + A_0(X_t)\,\mathrm{d}t, \quad X_0 = x. \tag{1}$$

Classical results:

- $A_0, A_1, \cdots, A_m$ are continuous and have linear growth $\Rightarrow$ SDE (1) has a weak solution;
- moreover, if the pathwise uniqueness holds for (1), then it has a unique strong solution.

Two extreme cases

(i) uniformly non-degenerate case: the matrix

$$a = (a_{ij})_{1\leq i,j\leq d} \geq C\,\mathrm{Id},$$

where $a_{ij} = \sum_{k=1}^m A_k^i A_k^j$ and $C > 0$;

(ii) completely degenerate case: $A_1 = \cdots = A_m = 0$, i.e. (1) reduces to an ODE.

Case (i): uniformly non-degenerate

- Stroock-Varadhan (1979): if $A_1, \cdots, A_m$ are bounded continuous and A_0 is bounded measurable, then SDE (1) has weak uniqueness;
- Veretennikov (1979): under stronger conditions, i.e. $A_1, \cdots, A_m$ are bounded Lipschitz, then SDE (1) has pathwise uniqueness;
- Krylov-Röckner (2005): unity diffusion ($m = d$ and $\sigma = (A_1, \cdots, A_d) = Id$) and $A_0 \in L^p_{loc}(\mathbb{R}^d)$ with $p > d + 2 \Rightarrow$ (1) has a unique strong solution;
- X. Zhang (2005): $A_1, \cdots, A_m$ are continuous and belong to $W^{1,2(d+1)}_{loc}(\mathbb{R}^d)$, $A_0 \in L^{2(d+1)}_{loc}(\mathbb{R}^d) \Rightarrow$ (1) has a unique strong solution.

The coefficients may depend on time.

Remark

Safonov (1999): even in the non-degenerate case, if the diffusion coefficients $A_1, \cdots, A_m$ lose the continuity, there are counterexamples for which the weak uniqueness does not hold.

Case (ii): completely degenerate

SDE (1) reduces to the ODE

$$\mathrm{d}X_t = A_0(X_t)\,\mathrm{d}t, \quad X_0 = x. \tag{2}$$

It is well know that if A_0 is not Lipschitz continuous, then ODE (2) may have no uniqueness of solutions, or it may have no solution at all.

Sobolev coefficient

- Cruzeiro (1983) first studied the existence of flow of measurable maps generated by Sobolev vector field b, provided that its gradient and divergence are exponentially integrable.
- Further developments: G. Peters (1996), Bogachev-Mayer Wolf (1999) and the book of Ustunel-Zakai (2000).

All these results require the exponential integrability of ∇A_0.

- DiPerna-Lions (1989) obtained similar result without the exponential integrability of ∇A_0. They proved this by studying the corresponding transport equation:

$$\frac{\partial u_t}{\partial t} = \langle A_0, \nabla u_t \rangle, \quad u|_{t=0} = u_0.$$

- Ambrosio (2004) generalized this result to BV vector fields.
- The extension to infinite dimensional case was done by Ambrosio-Figalli (2009) and Fang-Luo (2010).

Crippa-De Lellis (2008) obtained some new estimates on the perturbation of ODE (2), which allowed them to give a direct construction of the DiPerna-Lions flow.
More precisely, the absence of Lipschitz condition was filled by the following inequality: for $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$,

$$|f(x) - f(y)| \leq C_d\,|x-y|\,\big(M_R|\nabla f|(x) + M_R|\nabla f|(y)\big)$$

holds for $x, y \in N^c$ and $|x-y| \leq R$, where $M_R|\nabla f|$ is the local maximal function and N is a negligible set of $\mathbb{R}^d$; the classical moment estimate is replaced by estimating the quantity

$$\int_{B(R)} \log\left(\frac{|X_t(x) - \tilde X_t(x)|}{\sigma} + 1\right) \mathrm{d}x,$$

where $\sigma > 0$ is a small parameter.

X. Zhang (2010) developed this method for SDE and he obtained the stochastic flow of measurable maps generated by (1), mainly under the Sobolev regularity of the coefficients $A_0, A_1, \cdots, A_m$ and the boundedness of their divergences $\mathrm{div}(A_i)$.

Some notations

γ_d	the standard Gaussian measure on $\mathbb{R}^d$
$\mathbb{D}_1^p(\gamma_d)$	the first order Sobolev space
$\mathbb{D}_1^\infty(\gamma_d)$	$\bigcap_{p>1} \mathbb{D}_1^p(\gamma_d)$
$\delta(A)$	the divergence with respect to γ_d

Theorem (Fang-Luo-Thalmaier)

Assume that

(i) *the coefficients* $A_0, A_1, \ldots, A_m$ *have linear growth;*

(ii) *the diffusion coefficients* $A_1, \ldots, A_m$ *belong to* $\mathbb{D}_1^\infty(\gamma_d)$ *and the drift* $A_0 \in \mathbb{D}_1^q(\gamma_d)$ *for some* $q > 1$*;*

(iii) *there is* $\lambda_0 > 0$ *such that*

$$\int_{\mathbb{R}^d} \exp\left[\lambda_0\left(|\delta(A_0)| + \sum_{j=1}^m \left(|\delta(A_j)|^2 + |\nabla A_j|^2\right)\right)\right] \mathrm{d}\gamma_d < +\infty. \quad (3)$$

Then there is a unique stochastic flow of measurable maps $X\colon [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, *which solves* (1) *for almost all initial* $x \in \mathbb{R}^d$ *and the push-forward* $(X_t(w, \cdot))_{\#}\gamma_d$ *admits a density with respect to* γ_d, *which is in* $L^1 \log L^1$.

Ingredients of the proof

- a priori estimate of the density of the flow;
- a priori estimate of the perturbation of SDE (1).

Assume that $A_0, A_1, \cdots, A_m \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, then X_t is a stochastic flow of diffeomorphisms on $\mathbb{R}^d$. Kunita (1990) proved that $(X_t)_\# \gamma_d = K_t \gamma_d$, $(X_t^{-1})_\# \gamma_d = \tilde{K}_t \gamma_d$, and

$$\tilde{K}_t(x) = \exp\left(-\sum_{i=1}^m \int_0^t \delta(A_i)(X_s(x)) \circ \mathrm{d}w_s^i - \int_0^t \delta(\tilde{A}_0)(X_s(x))\, \mathrm{d}s \right), \tag{4}$$

where $\tilde{A}_0 = A_0 - \frac{1}{2} \sum_{i=1}^m \langle A_i, \nabla A_i \rangle$.

We do not have such a simple expression for K_t, but the following equality holds:

$$K_t(x) = \left[\tilde{K}_t\left(X_t^{-1}(x) \right) \right]^{-1}. \tag{5}$$

Estimate of the density

Using (4) and (5), we can get the a priori estimate below.

Theorem (Fang-Luo-Thalmaier)

For any $p > 1$,

$$\|K_t\|_{L^p(\mathbb{P}\times\gamma_d)} \leq \bigg[\int_{\mathbb{R}^d} \exp\bigg(pt\Big[2|\delta(A_0)| + \sum_{j=1}^m \big(|A_j|^2 + |\nabla A_j|^2 + 2(p-1)|\delta(A_j)|^2\big)\Big]\bigg) \mathrm{d}\gamma_d\bigg]^{\frac{p-1}{p(2p-1)}}.$$

This is an extension of Cruzeiro's estimate for the density of the flow associated to ODE (2).

Regularizing the coefficients

Let P_t be the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$:

$$P_t A(x) = \int_{\mathbb{R}^d} A\big(e^{-t}x + \sqrt{1-e^{-2t}}\, y\big)\mathrm{d}\gamma_d(y).$$

If A has linear growth, then there is $C > 0$ such that

$$\sup_{0<t\leq 1} |P_t A(x)| \leq C(1+|x|), \quad \text{for all } x \in \mathbb{R}^d.$$

Take a sequence of cut-off functions $\varphi_n \in C_c^\infty(\mathbb{R}^d, [0,1])$ satisfying

$$\varphi_n|_{B(n)} \equiv 1, \quad \mathrm{supp}(\varphi_n) \subset B(n+2) \quad \text{and} \quad \sup_{n\geq 1} \|\nabla\varphi_n\|_{L^\infty} \leq 1.$$

Now we define $A_i^n = \varphi_n P_{1/n} A_i$, then $A_i^n \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $0 \leq i \leq m$.

Approximating SDEs

Consider the sequence of SDEs

$$\mathrm{d}X_t^n = \sum_{i=1}^m A_i^n(X_t^n)\,\mathrm{d}w_t^i + A_0^n(X_t^n)\,\mathrm{d}t, \quad X_0^n = x.$$

Let K_t^n be the density of $(X_t^n)_{\#}\gamma_d$ with respect to γ_d, then by Theorem 3, there is T_0 small enough such that

$$\sup_{0\le t\le T_0}\sup_{n\ge 1}\|K_t^n\|_{L^p(\mathbb{P}\times\gamma_d)} \le \Lambda_{p,T_0} < +\infty. \tag{6}$$

Using the estimate (6), we can prove

Theorem

For $R > 0$ *and* $n, k \geq 1$, *define*

$$G_R^{n,k}(w) = \left\{ x \in \mathbb{R}^d : \sup_{0 \leq t \leq T_0} \left(|X_t^n(w,x)| \vee |X_t^k(w,x)| \right) \leq R \right\}.$$

Then there are $C_{T_0} > 0$ *and* $C_{d,q,R} > 0$ *s.t. for any* $\sigma > 0$,

$$\mathbb{E}\left[\int_{G_R^{n,k}} \log\left(\frac{\sup_{0\leq t\leq T_0} |X_t^n - X_t^k|^2}{\sigma^2} + 1 \right) \mathrm{d}\gamma_d \right] \leq C_{T_0} \Lambda_{p,T_0} \left(C_{d,q,R} + \frac{\Delta_{n,k}}{\sigma} + \frac{\Delta_{n,k}^2}{\sigma^2} \right),$$

where $\Delta_{n,k} = \|A_0^n - A_0^k\|_{L^q} + \left(\sum_{i=1}^m \|A_i^n - A_i^k\|_{L^{2q}}^2 \right)^{1/2}$.

Taking $\sigma = \Delta_{n,k}$, then the above theorem implies the family

$$I_{n,k} := \mathbb{E}\left[\int_{G_R^{n,k}} \log\left(\frac{\sup_{0\le t\le T_0}|X_t^n - X_t^k|^2}{\Delta_{n,k}^2} + 1\right) \mathrm{d}\gamma_d\right]$$

is bounded. Note that $\Delta_{n,k} \to 0$ as $n, k \to \infty$. By the linear growth of the coefficients, we can prove that for any $\alpha > 0$,

$$\lim_{n,k\to\infty} \mathbb{E}\int_{\mathbb{R}^d} \Big(\sup_{0\le t\le T_0} |X_t^n - X_t^k|^\alpha\Big) \mathrm{d}\gamma_d = 0.$$

That is, $\{X^n : n \ge 1\}$ is a Cauchy sequence in

$$L^\alpha(\Omega \times \mathbb{R}^d, C([0, T_0], \mathbb{R}^d)),$$

hence there exists $X : \Omega \times \mathbb{R}^d \to C([0, T_0], \mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \mathbb{E}\int_{\mathbb{R}^d} \Big(\sup_{0\le t\le T_0} |X_t^n - X_t|^\alpha\Big) \mathrm{d}\gamma_d = 0. \tag{7}$$

Now we can prove the main result on the small interval $[0, T_0]$. Using the estimate (6) and letting $n \to \infty$ in the equation below:

$$X_t^n = x + \sum_{i=1}^{m} \int_0^t A_i^n(X_s^n)\,\mathrm{d}w_s^i + \int_0^t A_0^n(X_s^n)\,\mathrm{d}s,$$

we obtain that for a.e. $x \in \mathbb{R}^d$, the following equality holds $\mathbb{P}$-almost surely:

$$X_t = x + \sum_{i=1}^{m} \int_0^t A_i(X_s)\,\mathrm{d}w_s^i + \int_0^t A_0(X_s)\,\mathrm{d}s, \quad \text{for all } t \in [0, T_0].$$

The absolute continuity can be proved as follows: $\forall\, \xi \in L^\infty(\Omega)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{array}{ccc}
\mathbb{E}\int_{\mathbb{R}^d} \xi \cdot \psi(X_t^n(x))\mathrm{d}\gamma_d(x) & = & \mathbb{E}\int_{\mathbb{R}^d} \xi \cdot \psi(y) K_t^n(y)\mathrm{d}\gamma_d(y) \\
\Big\downarrow{\scriptstyle (7)} & & \Big\downarrow{\scriptstyle (6)} \\
\mathbb{E}\int_{\mathbb{R}^d} \xi \cdot \psi(X_t(x))\mathrm{d}\gamma_d(x) & = & \mathbb{E}\int_{\mathbb{R}^d} \xi \cdot \psi(y) K_t(y)\mathrm{d}\gamma_d(y)
\end{array}$$

Therefore by the arbitrariness of ξ and ψ, we have $(X_t)_\# \gamma_d = K_t \gamma_d$.

Using the flow property of X_t^n and X_t, together with the following estimate

$$\sup_{0\leq t\leq T} \sup_{n\geq 1} \mathbb{E} \int_{\mathbb{R}^d} K_t^n \left|\log K_t^n\right| \mathrm{d}\gamma_d < +\infty,$$

we can prove the general case.

Thanks for your attention!