Moments, moderate and large deviations for a branching process in a random environment

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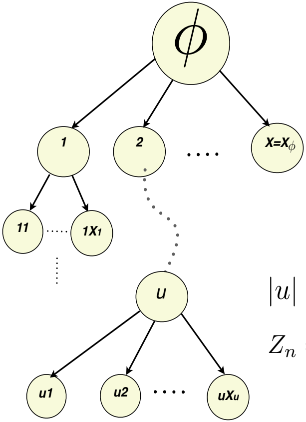
Description of BPRE

- Branching Process in a Random Environment (BPRE)



$$\xi = (\xi_n)_{(n\geq 0)} \quad i.i.d.$$

$$|u| = n, P_\xi(X_u = k) = p_k(\xi_n)$$

$$Z_n = {}^\sharp\{u : |u| = n\}$$ — the population size of n^{th} generation

Description of BPRE

Z_n – the population size of the nth generation,

X_u– the number of offspring of u.

By definition,

$$Z_0 = 1, \quad Z_{n+1} = \sum_{|u|=n} X_u, \quad (n \geq 0).$$

where given ξ , $\{X_u : |u| = n\}$ are conditionally independent of each other and have a common distribution $p(\xi_n) = \{p_k(\xi_n) : k \geq 0\}$.

Description of BPRE

- Quenched and annealed laws

  Let (Γ, P_ξ) be the probability space under which the process is defined when the environment ξ is fixed. As usual, P_ξ is called *quenched law*.

  The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, P)$, where $P = P_\xi \otimes \tau$ in the sense that for all measurable and positive g, we have

$$\int g dP = \int \int g(\xi, y) dP_\xi(y) d\tau(\xi),$$

  where τ is the law of the environment ξ. P is called *annealed law*. P_ξ may be considered to be the conditional probability of P given ξ.

Description of BPRE

- The martingale in BPRE
  Denote
$$m_n = \sum_k k p_k(\xi_n)$$
$$P_0 = 1, \qquad P_n = m_0 \cdots m_{n-1} \text{ for } n \geq 1.$$
  Then the normalized population size
$$W_n = \frac{Z_n}{P_n}$$
  is a nonnegative martingale and converges a.s. to a nonnegative random variable:
$$W = \lim_{n\to\infty} W_n \quad a.s.$$
  with $EW \leq 1$.

Description of BPRE

- Supercritical BRPE
  We consider the *supercritical* case where

$$E \log m_0 \in (0, \infty) \quad \text{and} \quad E \frac{Z_1}{m_0} \log^+ Z_1 < \infty.$$

For simplicity, let $p_k = p_k(\xi_0)$ and assume that

$$p_0 = 0 \quad a.s.,$$

Therefore

$$W > 0 \quad \text{and} \quad Z_n \to \infty \; a.s..$$

Description of BPRE

- Law of large numbers
  It is well known (Tanny(1977)) that

$$\lim_{n\to\infty} \frac{\log Z_n}{n} = E \log m_0 \quad a.s. \text{ (on } \{Z_n \to \infty\}).$$

We are interested in the asymptotic properties of the corresponding deviation probabilities.

Central Limit Theorem

- Central Limit Theorem
  We first remark that $\log Z_n$ satisfies the same central limit theorem as $\log P_n = \log m_0 + ... + \log m_{n-1}$:

Theorem 1 (Central Limit Theorem)

Assume that $E(\log m_0)^2 \in (0, \infty)$ and let $\sigma^2 = var(\log m_0)$. Then

$$\frac{\log Z_n - nE \log m_0}{\sqrt{n}\sigma} \to N(0, 1) \quad \text{in law.}$$

Large Deviations

The central limit theorem suggests that $\log Z_n$ and $\log P_n$ would satisfy the same large deviation principle. We shall prove this.

- Rate function
  Let
  $$\Lambda(t) = \log E m_0^t,$$
  and
  $$\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{xt - \Lambda(t)\}$$
  be the Fenchel-Legendre transform of Λ.

Large Deviations

We will use the following assumption:

Assumption (H)

There exist constants $\delta > 0$ and $A > A_1 > 1$ such that a.s.

$$A_1 \leq E_\xi Z_1, \qquad E_\xi Z_1^{1+\delta} \leq A^{1+\delta}.$$

Large Deviations

- Large Deviation Principle

Theorem 2 (Large Deviation Principle)

Assume (H). If $EZ_1^s < \infty$ for all $s > 1$ and $p_1 = 0$ *a.s.*, then for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^o} \Lambda^*(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log P\left(\frac{\log Z_n}{n} \in B\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{\log Z_n}{n} \in B\right) \\ &\leq -\inf_{x \in \bar{B}} \Lambda^*(x). \end{aligned}$$

Large Deviations

- Tail probabilities
  From Theorem 2, we obtain the following corollary:

Corollary (Bansaye and Berestycki (2009))

Under the conditions of Theorem 2, we have

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{\log Z_n}{n} \leq x\right) = -\Lambda^*(x) \quad \text{for } x < E \log m_0,$$
$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{\log Z_n}{n} \geq x\right) = -\Lambda^*(x) \quad \text{for } x > E \log m_0.$$

This result has been obtained by Bansaye and Berestycki in 2009. Our approach is different.

Moderate Deviations

- Moderate Deviation Principle
  - Large deviation: $\frac{\log Z_n - nE \log m_0}{n}$
  - Central limit theorem: $\frac{\log Z_n - nE \log m_0}{\sqrt{n}}$
  - Moderate deviation: $\frac{\log Z_n - nE \log m_0}{a_n}$

Let $\{a_n\}$ be a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty, \text{ as } n \to \infty.$$

Moderate Deviations

- Moderate Deviation Principle

Theorem 3 (Moderate Deviation Principle)

Assume (H) and write $\sigma^2 = var(\log m_0)$. Then for any measurable subset B of $\mathbb{R}$,

$$
\begin{aligned}
-\inf_{x\in B^o}\frac{x^2}{2\sigma^2} &\le \liminf_{n\to\infty}\frac{n}{a_n^2}\log P\left(\frac{\log Z_n - nE\log m_0}{a_n}\in B\right)\\
&\le \limsup_{n\to\infty}\frac{n}{a_n^2}\log P\left(\frac{\log Z_n - nE\log m_0}{a_n}\in B\right)\\
&\le -\inf_{x\in\bar{B}}\frac{x^2}{2\sigma^2}.
\end{aligned}
$$

Proof of Theorem 2 (LDP)

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  Notice that the Laplace transform of $\log Z_n$ is
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  Theorem 2 is a consequence of the *Gärtner-Ellis theorem* and the following result.

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Theorem 4 (Moments of Z_n)

Under certain moment conditions (*e.g.* the conditions of Theorem 2), we have

$$\lim_{n \to \infty} \frac{EZ_n^t}{(Em_0^t)^n} = C(t) \in (0, \infty), \quad \forall t \in \mathbb{R}.$$

This is an extension of a result of Ney and Vidyashankar (2003) on the Galton-Watson process.

Harmonic moments

- Harmonic moments
  To prove Theorem 4, we introduce a new BPRE and need a theorem about the harmonic moments of W:

Theorem 5 (Harmonic moments)

Assume (H).
(i) (General case) There exists a constant $a > 0$ such that

$$EW^{-a} < \infty.$$

(ii) (Special case) If $\|p_1\|_\infty := esssup\ p_1 < 1$, then $\forall a > 0$,

$$EW^{-a} < \infty \quad \text{if and only if} \quad Ep_1 m_0^a < 1.$$

Harmonic moments

Corollary (Critical value)

Assume (H) and $\|p_1\|_\infty < 1$. If $Ep_1 m_0^{a_0} = 1$, then

$$\begin{aligned} EW^{-a} &< \infty \quad \text{if } 0 < a < a_0, \\ EW^{-a} &= \infty \quad \text{if } a \geq a_0. \end{aligned}$$

Harmonic moments

Corollary (Critical value)

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Remark

Hambly (1992) proved that under some assumption similar to (H), the number $\alpha_0 := -\frac{E \log p_1}{E \log m_0}$ is the critical value for the a.s. existence of the quenched moments $E_\xi W^{-a} (a > 0)$. By Jensen's inequality, it is easy to see that $a_0 \leq \alpha_0$.

Proof of Theorem 5 (Harmonic moments)

- Proof of Theorem 5 (Harmonic moments)
  Set

$$\phi_\xi(t) = E_\xi e^{-tW} \quad \textit{and} \quad \phi(t) = E\phi_\xi(t)\ (t > 0).$$

Lemma A

Assume (H). Then there exist constants $\beta \in (0, 1)$ and $K > 0$ such that a.s.

$$\phi_\xi(t) \le \beta \quad \forall t \ge 1/K.$$

If additionally $\|p_1\|_\infty < 1$, then for some constants $a > 0$ and $C > 0$, a.s.

$$\phi_\xi(t) \le Ct^{-a} \quad \forall t \ge 1/K.$$

Proof of Theorem 5 (Harmonic moments)

- (ii) Special case where $\|p_1\|_\infty < 1$.
  - Necessity

$$W = \frac{1}{m_0} \sum_{i=1}^{Z_1} W_i^{(1)} \qquad a.s.,$$

where given ξ, $\left(W_i^{(1)}\right)_{i\geq 1}$ are conditionally i.i.d with $P_\xi(W_i^{(1)} \in \cdot) = P_{T\xi}(W \in \cdot)$. Since $P(Z_1 \geq 2) > 0$,

$$EW^{-a} > Em_0^a \left(W_1^{(1)}\right)^{-a} \mathbf{1}_{\{Z_1=1\}} = Ep_1 m_0^a EW^{-a}.$$

Therefore $Ep_1 m_0^a < 1$.

Proof of Theorem 5 (Harmonic moments)

- Sufficiency
  By Lemma A, $\forall \varepsilon > 0$, there exist a constant $t_\varepsilon > 0$ such that a.s.
  $$\phi_\xi(t) \leq \varepsilon \quad \forall t \geq t_\varepsilon.$$
  Notice that ϕ_ξ satisfies equation
  $$\phi_\xi(t) = f_0(\phi_{T\xi}(\frac{t}{m_0})),$$
  where $f_n(s) = \sum_{i=0}^{\infty} p_i(\xi_n)s^i, \; s \in [0,1]$. We have a.s.
  $$\phi_\xi(t) \leq (p_1 + (1-p_1)\varepsilon)\phi_{T\xi}(\frac{t}{m_0}) \quad \forall t \geq t_\varepsilon.$$

Proof of Theorem 5 (Harmonic moments)

Taking expectation, we obtain for $t \geq At_\varepsilon$,

$$\phi(t) \leq E(p_1 + (1 - p_1)\varepsilon)\phi(\frac{t}{m_0}) = p_\varepsilon E\phi(\tilde{A}_\varepsilon t),$$

where $p_\varepsilon = E(p_1 + (1 - p_1)\varepsilon) < 1$ and $\tilde{A}_\varepsilon$ is a positive random variable whose distribution is determined by

$$Eg(\tilde{A}_\varepsilon) = \frac{1}{p_\varepsilon} E(p_1 + (1 - p_1)\varepsilon) g(\frac{1}{m_0})$$

for all bounded and measurable function g.

Proof of Theorem 5 (Harmonic moments)

Since $Ep_1 m_0^a < 1$, We can take $a_1 > a$ such that $Ep_1 m_0^{a_1} < 1$. Take $\varepsilon > 0$ small enough such that

$$p_\varepsilon E\tilde{A}_\varepsilon^{-a_1} = E(p_1 + (1 - p_1)\varepsilon)m_0^{a_1} < 1.$$

Then by Liu (2001),

$$\phi(t) = O(t^{-a_1}), \quad \text{so that } \; EW^{-a} < \infty.$$

- (i) General case
  Notice that $\phi_\xi(t) \leq \beta$ *a.s.* for $t \geq t_\beta = \frac{1}{K}$. It suffices to repeat the proof of sufficiency of (ii) with β in place of ε.

Proof of Theorem 4 (Moments of Z_n)

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Under certain moment conditions (*e.g.* the conditions of Theorem 2), we have

$$\lim_{n\to\infty} \frac{EZ_n^t}{(Em_0^t)^n} = C(t) \in (0,\infty), \quad \forall t \in \mathbb{R}.$$

- Proof

  Denote the distribution of ξ_0 by τ_0. Fix $t \in \mathbb{R}$ and define a new distribution $\tilde{\tau}_0$ as

$$\tilde{\tau}_0(dx) = \frac{m(x)^t \tau_0(dx)}{Em_0^t},$$

where $m(x) = E[Z_1|\xi_0 = x] = \sum_{i=0}^{\infty} ip_i(x)$.

Proof of Theorem 4 (Moments of Z_n)

Consider the new BPRE whose environment distribution is $\tilde{\tau} = \tilde{\tau}_0^{\otimes \mathbb{N}}$ instead of $\tau = \tau_0^{\otimes \mathbb{N}}$. The corresp. total probability and expectation are denoted by $\tilde{P} = P_\xi \otimes \tilde{\tau}$ and $\tilde{E}$.

Then

$$\frac{EZ_n^t}{\left(Em_0^t\right)^n} = \tilde{E} W_n^t.$$

We distinguish three cases: $t \in (0, 1)$, $t > 1$ and $t < 0$. For each case, under certain moment conditions,

$$\lim_{n \to \infty} \tilde{E} W_n^t = \tilde{E} W^t \in (0, \infty).$$

Proof of Theorem 3 (MDP)

- Proof of Theorem 3 (MDP)
  Similar to the proof of Theorem 2 (LDP), the proof of Theorem 3 is a combination of the *Gärtner-Ellis theorem* and the following result.

Theorem 6

Assume (H). We have

$$\lim_{n\to\infty} \frac{\log EZ_n^{\frac{a_n}{n}t}}{\log EP_n^{\frac{a_n}{n}t}} = 1 \qquad \forall t \neq 0.$$

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Thank you !

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