Decay Property of Markov Branching Processes with Immigration and Disaster

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1 Background

- Definition of Decay parameter

Let $\mathbf{E}$ be a countable set.

$Q = (q_{ij}; i, j \in \mathbf{E})$ be a stable q-matrix.

$(p_{ij}(t); i, j \in \mathbf{E}$ is the Feller minimal Q-process.

C is a communicating class of $\mathbf{E}$ and

$$\lim_{t\to\infty} p_{ij}(t) = 0, \quad i, j \in C.$$

By Kingman (1936), there exists a number $\lambda_C \geq 0$ such that for all $i, j \in C$,

$$\frac{1}{t}\log p_{ij}(t) \to -\lambda_C \text{ as } t \to \infty$$

λ_C is called the decay parameter for C.

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On the other hand, let

$$\mu_{ij} = \inf\{\lambda \geq 0 : \int_0^\infty e^{\lambda t} p_{ij}(t)dt = \infty\}$$
$$= \sup\{\lambda \geq 0 : \int_0^\infty e^{\lambda t} p_{ij}(t)dt < \infty\}.$$

It is easily seen that μ_{ij} does not depend on $i, j \in C$, the common value is denoted by μ. Moreover,

$$\lambda_C = \mu.$$

(see, for example, Pollett (2006)).

- Definition of λ_C-recurrence

λ_C-recurrent: $\int_0^\infty e^{\lambda_C t} p_{ii}(t)dt = +\infty, \ \forall i \in C$

λ_C-transient: $\int_0^\infty e^{\lambda_C t} p_{ii}(t)dt < +\infty, \ \forall i \in C$

Positively λ_C-recurrent: $\lim_{t\to\infty} e^{\lambda_C t} p_{ii}(t)dt > 0, \ \forall i \in C$

λ_C-subinvariant measure $(m_k; k \in C)$ for Q:

$$\sum_{k\in C} m_k q_{kj} \le -\lambda_C m_j,\ m_j > 0\,\forall j \in C.$$

It is called λ_C-invariant if the equality holds.

λ_C-subinvariant vector $(x_k; k \in C)$ for Q:

$$\sum_{j\in C} q_{ij} x_j \le -\lambda_C x_i,\ x_i > 0,\ \forall i \in C.$$

It is called λ_C-invariant if the equality holds.

Similarly, one can define λ_C-(sub)invariant measure/vector for $P(t)$.

- Problems:

▶ $\lambda_C = ?$;

▶ The λ_C-recurrency of the process.

- Known progress:

(i) Finite Markov chains.

(ii) BDP(Chen M.F.).

Special BDP: $q_{i\ i-1} = a$, $q_{i\ i+1} = b$, then $\lambda_C = (\sqrt{a} - \sqrt{b})^2$.

(iii) MBP: $q_{ij} = ib_{j-i+1}$, then $\lambda_C = -B'(q)$ where

$$B(s) = \sum_{j=0}^{\infty} b_j s^j$$

and q is the smallest nonnegative root of $B(s) = 0$.

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(iv) Stopped $M^X/M/1$ queue (Li and Chen, 2008):

$$q_{ij} = \begin{cases} b_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda_C = \sup\{\lambda \geq 0 : B(s) + \lambda s = 0 \text{ has a root in } (0, +\infty)\}$$

where $B(s) = \sum_{k=0}^{\infty} b_k s^k$.

(v) Controlled $M^X/M/1$ queue (Li and Chen, 2009):

$$q_{ij} = \begin{cases} h_j, & \text{if } i = 0,\ j \geq 0 \\ b_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda_C$$

$$= \sup\{\lambda \geq 0 : B(s) + \lambda s \leq 0,\ \lambda + H(s) \leq 0 \text{ has a root in } (0,1)\}$$

where $B(s) = \sum_{k=0}^{\infty} b_k s^k$ and $H(s) = \sum_{k=0}^{\infty} h_k s^k$.

- Description of the models in this talk

Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be defined as follows:

$$q_{ij} = \begin{cases} ib_{j-i+1} + a_{j-i}, & \text{if } i \geq 0, j \geq i \\ b_0, & \text{if } i \geq 1, j = (i-1) \vee 0 \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where

$$\begin{cases} b_j \geq 0 \; (j \neq 1), \; 0 < \sum_{j \neq 1} b_j \leq -b_1 < \infty \\ a_j \geq 0 \; (j \neq 0), \; 0 < \sum_{j=1} a_j \leq a_0 < \infty. \end{cases} \tag{2}$$

Q is called a BI q-matrix.

Definition 1. *A Markov branching process with immigration (MBPI) is a continuous Markov chain on the state space* $\mathbf{Z}_+$ *whose transition function* $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ *satisfies*

$$P'(t) = P(t)Q \tag{3}$$

where Q *is a BI q-matrix defined in (1)–(2).*

Define

$$A(s) = \sum_{k=0}^{\infty} a_k s^k \quad \text{and} \quad B(s) = \sum_{k=0}^{\infty} b_k s^k. \tag{4}$$

Proposition 1. (i) $A(s) < 0$ for all $s \in [-1, 1)$ and

$$A(s) \upuparrows A(1) \leq 0 \text{ as } s \uparrow 1.$$

(ii) $B(s)$ is convex on $[0, 1]$ and $B(s) = 0$ possesses a smallest nonnegative root ρ. Moreover, $\rho = 1$ iff $B(1) = 0$ and $B'(1) \leq 0$.

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It is clear that Q is conservative iff $B(1) = A(1) = 0$.

For Q being conservative case:

The recurrence and hitting time properties, Sevast'yanov(1957), Zubkov(1972) and Vatutin(1977).

Li and Chen(2006) considered a more general model in which $q_{0j} = a_j \; (j \geq 0)$ are replaced by arbitrary rates $h_j \; (j \geq 0)$.

Proposition 2.*(Li & Chen 2006) Suppose that* $A(1) = B(1) = 0$. *Then*

(i) Q *is regular iff either* $B'(1) < \infty$ *or* $B'(1) = \infty$ *together with* $\int_\varepsilon^1 \frac{ds}{-B(s)} = \infty$ *for some (equivalently for all)* $\varepsilon \in (\rho, 1)$ *where* $\rho < 1$ *is the smallest nonnegative root of* $B(s) = 0$.

(ii) *The MBPI is recurrent iff* $B'(1) \le 0$ *and* $J = +\infty$ *where*

$$J := \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy. \tag{5}$$

Moreover, the MBPI is positive recurrent iff $B'(1) \le 0$ *and*

$$\int_0^1 \frac{-A(s)}{B(s)} ds < \infty. \tag{6}$$

If $B(1) < 0$ or $A(1) < 0$, then we can consider another q-matrix $\widetilde{Q} = (\widetilde{q}_{ij}; i, j \in \mathbf{Z}_+ \cup \{\Delta\})$ (where Δ is an added state):

$$\widetilde{q}_{ij} = \begin{cases} 0, & \text{if } i = \Delta, j \in \mathbf{Z}_+ \cup \{\Delta\} \\ -iB(1) - A(1), & \text{if } i \geq 0, j = \Delta \\ ib_{j-i+1} + a_{j-i}, & \text{if } i \geq 0, j \geq i \\ ib_0, & \text{if } i \geq 0, j = i-1 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$\widetilde{Q}$ is called a BID q-matrix. The corresponding process is called an MBPID.

This talk is concentrated on the transiency property of Q(or) $\widetilde{Q}$-process.

2 Preliminary

In order to find the exact value of decay parameter λ_Z and discuss the λ_Z-recurrence property, we need some preparation.

Lemma 1. *There always exists only one MBPID which satisfies the Kolmogorov forward equations.*

Lemma 2. *Let Q be defined in $(1)-(2)$ and $P(t)=(p_{ij}(t); i,j\geq 0)$ be the Feller minimal Q-process. Then for any $i\geq 0$ and $|s|<1$,*

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s)\sum_{j=1}^{\infty} p_{ij}(t)js^{j-1} + A(s)\sum_{j=0}^{\infty} p_{ij}(t)s^j. \tag{8}$$

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Lemma 3. *Let $P(t) = (p_{ij}(t); i, j \geq 0)$ be a transition function. Then the following two statements are equivalent.*

(i) *$P(t)$ is the Feller minimal Q-function, where Q takes the form of (1) − (2).*

(ii) *For any $i \geq 0,\ t \geq 0,\ s \in [-1, 1]$, we have*

$$F_i(t, s) = F_0(t, s) \cdot \sum_{j=0}^{\infty} p^*_{ij}(t) s^j \tag{9}$$

where $F_i(t, s) = \sum_{j=0}^{\infty} p_{ij}(t) \cdot s^j\ \ (i \geq 0,\ s \in [-1, 1])$ and $P^*(t) = (p^*_{ij}(t); i, j \geq 0)$ is a Markov branching process whose q-matrix Q^* (may not be conservative) is given by

$$q^*_{ij} = \begin{cases} ib_{j-i+1}, & \text{if } i \geq 0, j \geq i-1 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

where $\{b_j; j \geq 0\}$ is the same as given in (2).

Sketch of the proof. (i)⇒ (ii). By Lemma 2,

$$\frac{\partial F_i(t,s)}{\partial t} = B(s)\cdot\frac{\partial F_i(t,s)}{\partial s} + A(s)F_i(t,s) \qquad (11)$$

where $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$. Let Q^* be given by (10) and $P^*(t) = (p^*_{ij}(t); i,j \geq 0)$ be the minimal Q^*-function and define $\hat{P}(t)$ by $\hat{p}_{ij}(t) = \sum_{k=0}^{j} p_{0k}(t)p^*_{kj}(t)$. Then $\hat{P}'(t) = \hat{P}(t)Q$. By Lemma 1, we must have $\hat{P}(t) = P(t)$.

(ii)⇒(i). Note that for any $i,j \geq 0$ and $0 < s < 1$,

$$p_{ij}(t)s^j \leq F_0(t,s)\Big(\sum_{k=0}^{\infty} p^*_{1k}(t)s^k\Big)^i.$$

which leads $\lim_{i\to\infty} p_{ij}(t) = 0$. Therefore, by Reuter and Riley [7] or Anderson [1], $P(t)$ is the Feller minimal Q-function. □

3 Conclusions

Theorem 1 . *Let* Q *be defined as in* $(1)-(2)$ *and* $P(t) = (p_{ij}(t); i, j \geq 0)$ *be the Feller minimal* Q*-function. Then*

$$\lambda_Z = -A(\rho)$$

where ρ *is the smallest nonnegative root of* $B(s) = 0$*. In particular,* $\lambda_Z = 0$ *if and only if* $\rho = 1$ *and* $A(1) = 0$*, i.e., if and only if* Q *is conservative and* $B'(1) \leq 0$.

Sketch of the proof.

(i) $\lambda_Z \geq -A(\rho)$. $(\rho^k; k \geq 0)$ is a $-A(\rho)$-invariant vector for Q.

(ii) $\lambda_Z \leq -A(\rho)$.

(a) Case 1: $\rho < 1$. define

$$\bar{p}_{ij}(t) = e^{-A(\rho)t} p_{ij}(t) \rho^{j-i}, \quad i, j \geq 0, t \geq 0. \tag{12}$$

Then $\bar{P}(t) = (\bar{p}_{ij}(t); i, j \geq 0)$ is a standard and honest transition function. Its q-matrix $\bar{Q} = (\bar{q}_{ij}; i, j \geq 0)$ is given by

$$\bar{q}_{ij} = \begin{cases} i\bar{b}_{j-i+1} + \bar{a}_{j-i}, & \text{if } i \geq 0, j \geq i \\ i\bar{b}_0, & \text{if } i \geq 1, j = i-1 \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

where $\bar{a}_j = a_j \rho^j - A(\rho)\delta_{0j}$ $(j \geq 0)$ and $\bar{b}_j = b_j \rho^j$ $(j \geq 0)$. Applying Proposition 1 to $\bar{Q}$ will imply the $\bar{Q}$-process is recurrent, Hence $\lambda_Z = -A(\rho)$.

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(b) Case 2: $\rho = 1$. For any $\varepsilon > 0$, define

$$q_{ij}^{(\varepsilon)} = \begin{cases} ib_{j-i+1}^{(\varepsilon)} + a_{j-i}, & \text{if } i \geq 0, j \geq i \\ ib_0^{(\varepsilon)}, & \text{if } i \geq 0, j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

where $b_k^{(\varepsilon)} = b_k - \varepsilon\delta_{k1}$. Then $Q^{(\varepsilon)} = (q_{ij}^{(\varepsilon)}; i, j \geq 0)$ is a nonconservative BID q-matrix. Let $P^{(\varepsilon)}(t) = (p_{ij}^{(\varepsilon)}(t); i, j \geq 0)$ be the minimal $Q^{(\varepsilon)}$-function. It can be proved that $p_{ij}^{(\varepsilon)}(t) \leq p_{ij}(t)$. However, $P^{(\varepsilon)}(t)$ has the decay parameter $\lambda_Z^{(\varepsilon)} = -A(\rho_\varepsilon)$ and hence $\lambda_Z \leq \lambda_Z^{(\varepsilon)} = -A(\rho_\varepsilon)$. Now, letting $\varepsilon \downarrow 0$ yields that $\lambda_Z \leq -A(1)$. $\square$

Theorem 2. *Let Q be defined as in $(1)-(2)$ and $P(t) = (p_{ij}(t); i, j \geq 0)$ be the Feller minimal Q-function and λ_Z be the decay parameter of $P(t)$ on $\mathbf{Z}_+$.*

(i) *If $B'(1) > 0$ then $P(t)$ is λ_Z-positive.*

(ii) *If $B'(1) \leq 0$ then $P(t)$ is λ_Z-recurrent if and only if*

$$\widetilde{J} = \int_0^1 \frac{1}{B(s)} e^{\int_0^s \frac{A(y)-A(1)}{B(y)} dy} ds = +\infty. \tag{14}$$

Moreover, $P(t)$ is λ_Z-positive if and only if

$$\int_0^1 \frac{A(1)-A(y)}{B(y)} dy < \infty. \tag{15}$$

Sketch of the proof. Note that if $\rho < 1$ then $(\bar{p}_{ij}(t); i, j \geq 0)$ is recurrent. Also note $\bar{B}(1) = \bar{A}(1) = 0$ together with $\bar{B}'(1) < 0$ and $\bar{A}'(1) < \infty$, applying Proposition 2 to $\bar{Q}$ will yield (i).

Secondly, suppose that $B'(1) \leq 0$ and thus by Theorem 1 we have $\lambda_Z = -A(1)$. Define

$$\bar{p}_{ij}(t) = e^{\lambda_Z t} p_{ij}(t), \quad i, j \geq 0, t \geq 0.$$

Apply Proposition 2 and Theorem 1 to $(\bar{p}_{ij}(t))$, we get (ii). $\square$

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Theorem 3. **(i)** *there exists a* λ_Z*-invariant measure* $(m_i; i \geq 0)$ *for* Q *on* $\mathbf{Z}_+$*, which is unique up to constant multiples. Moreover,* $M(s) = \sum_{i=0}^{\infty} m_i s^i$ *is given by*

$$M(s) = m_0 e^{\int_0^s \frac{A(\rho)-A(y)}{B(y)} dy}, \quad |s| < \rho \tag{16}$$

where $m_0 > 0$ *is a constant.*

(ii) $(m_i; i \geq 0)$ *is also a* λ_Z*-invariant for* $P(t)$*.*

(iii) $M(1) = \sum_{i=0}^{\infty} m_i < \infty$ *if and only if* $B'(1) \leq 0$ *and*

$$\int_0^1 \frac{A(1) - A(y)}{B(y)} dy < \infty.$$

(iv) $(\rho^k; k \geq 0)$ *is a* λ_Z*-invariant vector for* $P(t)$ *on* $\mathbf{Z}_+$*. Moreover, if* $B'(1) > 0$ *or* $B'(1) \leq 0$ *with* (14) *holds, then* $(\rho^k; k \geq 0)$ *is the unique (up to constant multiples)* λ_Z*-invariant vector for* $P(t)$ *on* $\mathbf{Z}_+$*.*

4 Applications

• Q being conservative.

Theorem 4. *The minimal Q-function is the unique MBPID. Moreover,*

(i) if $B'(1) \leq 0$, then $\lambda_Z = 0$ and the MBPID is 0-recurrent iff $B'(1) \leq 0$ and $J = +\infty$ where J is given in (5).

(ii) If $B'(1) > 0$ then $\lambda_Z = -A(\rho) > 0$. Also, the MBPID is positively λ_Z-recurrent and there exists a unique (up to constant multiples) λ_Z-invariant measure $(m_i; i \geq 0)$ whose generating function $M(s) = \sum_{i=0}^{\infty} m_i s^i$ is given by

$$M(s) = m_0 \exp\{\int_0^s \frac{A(\rho) - A(y)}{B(y)} dy\}, \quad |s| < \rho.$$

Furthermore, this λ_Z-invariant measure is not summable and thus there does not exist any quasi-stationary distribution.

• Q being not conservative.

Theorem 5 . (i) *The Feller minimal $\tilde{Q}$-function is the unique $\tilde{Q}$-function satisfying Kolmogorov forward equation.*

(ii) *$\tilde{Q}$ is not regular iff $B(1) = 0$ (thus $A(1) < 0$), $B'(1) = +\infty$ and $\int_{\varepsilon}^{1} \frac{ds}{-B(s)} < +\infty$ for some (equivalently for all) $\varepsilon \in (\rho, 1)$ where $\rho < 1$.*

(iii) *If $\tilde{Q}$ is regular, then $a_{i\Delta} = 1$ $(i \geq 0)$. If $\tilde{Q}$ is not regular, then*

$$a_{i\Delta} = A(1) \cdot \int_{\rho}^{1} \frac{y^i}{B(y)} e^{-\int_y^1 \frac{A(x)}{B(x)} dx} dy \quad \text{and} \quad a_{i\infty} = 1 - a_{i\Delta} \quad (17)$$

where $a_{i\Delta}$ and $a_{i\infty}$ are the extinction and explosion probability of the Feller minimal $\tilde{Q}$-process, respectively.

References

[1] Anderson,W. (1991). *Continuous-Time Markov Chains: An Applications-Oriented Approach*. Springer-Verlag, New York.

[2] Athreya, K. B. and Jagers, P. (1996). *Classical and Modern Branching Processes*. Springer, Berlin.

[3] Chen, A.Y., Li, J.P. and Ramesh, N.I. (2005). Uniqueness and Extinction of Weighted Markov Branching Processes. *Methodology and Computing in Applied Probability.* 7, 489-516.

[4] Kingman,J.F.C. (1963) The exponential decay of Markov transition probability. *Proc. London Math. Soc.*, 13, 337-358.

[5] Li, J.P. and Chen, A.Y. (2006) Markov Branching Processes with Immigration and Resurrection. *Markov Processes and Related Fields* , 12, 139-168.

[6] Pollett,P.K.(1986) On the equivalence of mu-invariant measures for the minimal process and its q-matrix. *Stochastic Proc. Appl.* , 22, 203-221.

[7] Reuter, G.E.H. and Riley, P.W. (1972) The Feller property for Markov semigroups on a countable state space. *J. London Math. Soc.* , (2)5, 267-275.

[8] Sevast'yanov, B.A. (1957) Limit theorems for branching stochastic processes of special form. *Theory Prob. Applications.* , 2(3), 321-331.

[9] Vatutin, V.A. (1977) A conditional limit theorem for a critical branching process with immigration. *TMathematical Notes.*, 21, 405-411.

[10] Yang,X.Q. (1990). *The Construction Theory of Denumerable Markov Processes*. Wiley, New York.

[11] Zubkov, A.M. (1972) The life-periods of a branching process with immigration. *Theory Prob. Applications*, 17(1), 174-183.

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