A Geometric Process Maintenance Model

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Outline

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Definition of GP

- In maintenance problem, the perfect repair model and minimal repair model are popular. However, as most systems are deteriorating, the successive operating times of a system after repair will be decreasing, while the consecutive repair times after failure will be increasing. Thus, a more direct approach is to introduce a monotone process model. To do this, Lam (1988a, b) first introduced the GP that is a simple monotone process.

- **Definition** A sequence of nonnegative random variables $\{X_n, n = 1, 2, \ldots\}$ is said to be a geometric process (GP), if they are independent and the distribution function of X_n is given by $F(a^{n-1}x)$ for $n = 1, 2, \ldots$, where $a > 0$ is called the ratio of the GP.

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Property of GP

- A GP is (stochastically) increasing if the ratio $0 < a \leq 1$; it is (stochastically) decreasing if the ratio $a \geq 1$. A GP will become a renewal process if the ratio $a = 1$. Therefore, GP is a simple monotone process and is a generalization of the renewal process.
- Assume that $\{X_n, n = 1, 2, \ldots\}$ is a GP with ratio a. Let $E(X_1) = \lambda$ and $\mathrm{Var}(X_1) = \sigma^2$. Then

$$E[X_n] = \frac{\lambda}{a^{n-1}}, \tag{1}$$

and

$$\mathrm{Var}(X_n) = \frac{\sigma^2}{a^{2(n-1)}}. \tag{2}$$

Thus, a, λ and σ^2 are three important parameters of the GP.

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Model Assumptions

- **Assumption 1.** At the beginning, a new system is installed. Whenever the system fails, it will be repaired. A replacement policy N is applied by which the system is replaced by a new, identical one at the time following the Nth failure.
- **Assumption 2.** Let X_1 be the system operating time after the installation or a replacement. In general, for $n > 1$, let X_n be the system operating time after the $(n-1)$th repair, then $\{X_n, n = 1, 2, \ldots\}$ form a GP with $E(X_1) = \lambda > 0$ and ratio a . Moreover, let Y_n be the system repair time after the nth failure, then $\{Y_n, n = 1, 2, \ldots\}$ constitute a GP with $E(Y_1) = \mu \geq 0$ and ratio b. Let the replacement time be Z with $E(Z) = \tau$.

Model Assumptions

- **Assumption 3.** The operating reward rate is r, the repair cost rate is c. The replacement cost comprises two parts: one part is the basic replacement cost R, and the other part is proportional to the replacement time Z at rate c_p.
  Besides, an additional assumption is made from one of the following two assumptions.
- **Assumption 4.** $a \geq 1$ and $0 < b \leq 1$.
- **Assumption 4'.** $0 < a \leq 1$ and $b \geq 1$ except the case $a = b = 1$.
- Then under Assumptions 1-4, the GP maintenance model is a model for a deteriorating system. However, under Assumptions 1-3 and 4', the GP maintenance model is a model for an

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Average Cost

- Now, we say that a cycle is completed if a replacement is completed. Thus a cycle is actually a time interval between the installation of a system and the first replacement or a time interval between two consecutive replacements. Then, the successive cycles together with the costs incurred in each cycle will constitute a renewal reward process.
- The long-run average cost per unit time (or simply the average cost) is given by

$$\frac{\text{Expected cost incurred in a cycle}}{\text{Expected length of a cycle}}. \tag{3}$$

Average Cost

- Consequently, under Assumptions 1-3 and using policy N, the average cost is given by

$$\begin{aligned} C(N) &= \frac{E(c\sum_{k=1}^{N-1} Y_k - r\sum_{k=1}^{N} X_k + R + c_p Z)}{E(\sum_{k=1}^{N} X_k + \sum_{k=1}^{N-1} Y_k + Z)} \\ &= \frac{c\mu\sum_{k=1}^{N-1}\frac{1}{b^{k-1}} - r\lambda\sum_{k=1}^{N}\frac{1}{a^{k-1}} + R + c_p\tau}{\lambda\sum_{k=1}^{N}\frac{1}{a^{k-1}} + \mu\sum_{k=1}^{N-1}\frac{1}{b^{k-1}} + \tau} \qquad (4) \\ &= A(N) - r, \end{aligned}$$

where

Average Cost

- $$A(N) = \frac{(c+r)\mu \sum_{k=1}^{N-1} \frac{1}{b^{k-1}} + R + c_p\tau + r\tau}{\lambda \sum_{k=1}^{N} \frac{1}{a^{k-1}} + \mu \sum_{k=1}^{N-1} \frac{1}{b^{k-1}} + \tau}. \tag{5}$$

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Key Lemma

- Now, our objective is to determine an optimal replacement policy N^* for minimizing $C(N)$ or $A(N)$.
- Consider

$$A(N+1) - A(N) = \frac{(c+r)\mu\{\lambda(\sum_{k=1}^{N} a^k - \sum_{k=1}^{N-1} b^k) + \tau a^N\} - (R + c_p\tau + r\tau)(\lambda b^{N-1} + \mu a^N)}{a^N b^{N-1}[\lambda \sum_{k=1}^{N} \frac{1}{a^{k-1}} + \mu \sum_{k=1}^{N-1} \frac{1}{b^{k-1}} + \tau][\lambda \sum_{k=1}^{N+1} \frac{1}{a^{k-1}} + \mu \sum_{k=1}^{N} \frac{1}{b^{k-1}} + \tau]}.$$

Key Lemma

- Define an auxiliary function

$$
\begin{aligned}
g(N) &= g(N, a, b, \lambda, \mu, \tau, r, c, R, c_p) \\
&= \frac{(c+r)\mu\{\lambda(\sum_{k=1}^{N} a^k - \sum_{k=1}^{N-1} b^k) + \tau a^N\}}{(R + c_p\tau + r\tau)(\lambda b^{N-1} + \mu a^N)}. \qquad (6)
\end{aligned}
$$

- **Lemma 1.**

$$
A(N+1) \gtreqqless A(N) \iff g(N) \gtreqqless 1.
$$

Key Lemma

- Note that all the results above are developed under Assumptions 1-3 only. Therefore, the above results hold for the models of a deteriorating system and of an improving system.

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The model under Assumptions 1-4

- The model is a model for a deteriorating system. Now, function $g(N)$ is nondecreasing in N.

- **Theorem 1.** An optimal replacement policy N_d^* for the deteriorating system is determined by

$$N_d^* = \min\{N \mid g(N) = g(N, a, b, \lambda, \mu, \tau, r, c, R, c_p) \geq 1\}. \qquad (7)$$

  The optimal replacement policy N_d^* is unique if and only if $g(N_d^*) > 1$.

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The model under Assumptions 1-3 and 4’

- The model is a model for an improving system. Now, function $g(N)$ is decreasing in N.
- **Theorem 2.** Under Assumptions 1-3 and 4’, policy $N_i^* = \infty$ is the unique optimal policy for the improving system.
- Intuitively, it is a general knowledge that the older the improving system is, the better the system is. This means that we shall repair the system forever when it fails without replacement. Clearly, Theorem 2 agrees with this general knowledge. See Lam (2003, 2007a) for the proof of Theorems 1 and 2.

- Here we just study the monotonicity of the optimal policy for a deteriorating system, and use N^* to denote its optimal policy. It follows from g that N^* will be a function of $a, b, \lambda, \mu, \tau, r, c, R$ and c_p and can be denoted by

$$N^* = N^*(a, b, \lambda, \mu, \tau, r, c, R, c_p).$$

- **Lemma 2.** Auxiliary function g and optimal policy N^* possess an opposite monotonicity property in each parameter of $a, b, \lambda, \mu, \tau, r, c, R$ and c_p.

- **Theorem 3.**

$$(1) \quad N^*(a,b,\lambda,\mu,\tau,r,c,R,c_p) \le N^*(a,b',\lambda,\mu,\tau,r,c,R,c_p), \forall b < b'.$$

$$(2) \quad N^*(a,b,\lambda,\mu,\tau,r,c,R,c_p) \ge N^*(a,b,\lambda,\mu',\tau,r,c,R,c_p), \forall \mu < \mu'$$

$$(3) \quad N^*(a,b,\lambda,\mu,\tau,r,c,R,c_p) \ge N^*(a,b,\lambda,\mu,\tau,r,c',R,c_p), \forall c < c'.$$

$$(4) \quad N^*(a,b,\lambda,\mu,\tau,r,c,R,c_p) \le N^*(a,b,\lambda,\mu,\tau,r,c,R',c_p), \forall R < R'$$

$$(5) \quad N^*(a,b,\lambda,\mu,\tau,r,c,R,c_p) \le N^*(a,b,\lambda,\mu,\tau,r,c,R,c_p^{'}), \forall c_p < c_p^{'}$$

- The results are consistent with our practical experience. For example, as the replacement cost R increases while the other parameters keep unchanged, one would delay the replacement for saving expenses. Similarly, as the repair cost rate c raises with the others fixed, one would prefer to replace the system earlier for reducing expenditure. In other words, optimal policy N^* is increasing in R but decreasing in c.

- See Lam (2003, 2007a) for the proof of Theorem 3. Besides, in Lam (2003, 2007a), he also discuss the monotonicity of N^* in other parameters. However, the monotonicity of N^* in parameter a is still unknown. This is an open problem leaving for further research.

- In most maintenance models so far, we only consider a two-state system, up and down states say. In practice, a system may have more than 2 states. Thus, Lam et al. (2002) and Lam (2005) studied a monotone process model for a multistate system.
- In the GP maintenance model, we just studied the model for a system that is deteriorating due to the internal cause, the ageing effect or accumulated wear say. In practical problem, sometimes the deteriorating may be caused by the external environment. Lam and Zhang (2003) considered a GP maintenance model under a random environment. Moreover, Tang and Lam (2006) and Lam (2009) studied a GP δ-shock maintenance model.

- In practice, many systems demonstrate that their failure rate has a bathtub curve shape. As a result, in the early stage, the failure rate is decreasing, during the middle stage, the failure rate may be approximately a constant, and in the late stage, the failure rate is increasing. Thus, the system will be improving in the early stage, steady in the middle stage, and deteriorating in the late stage. For modeling such a system, a threshold GP model should be introduced. Lam (2007a) introduced a threshold GP and studied a threshold GP maintenance model.

- To improving the system reliability, a preventive repair is frequently adopted. Accordingly, Lam (2007b) considered a GP maintenance model with preventive repair.

- For understanding the probability and statistical theory of GP, and more applications of GP, see Lam (2007a) for a comprehensive reference.

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Thank You !