Some central limit theorems for the Brownian local time in $L^p(R)$

Yaozhong Hu
University of Kansas

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David Nualart

Based on the following papers

Hu Y., Nualart, D.

Stochastic integral representation of the L^2 modulus of Brownian local time and a central limit theorem.

Electron. Commun. Probab. 14 (2009), 529 - 539.

Hu Y., Nualart, D.

Central limit theorem for the modulus of continuity of the Brownian local time in $L^3(R)$

Submitted.

Hu Y., Nualart, D.

Central limit theorem for the p-integrated moment of Brownian local time increments

In progress.

Outline of The Talk

1. Background

2. Main results

3. Two tools

4. General case

1. Background

Let $\{B_t, t \geq 0\}$ be a standard one-dimensional Brownian motion.

$\{L_t^x, t \geq 0, x \in R\}$ a continuous version of its local time.

$$\int_0^t f(B_s)ds = \int_{\mathbf{R}} f(x) L_t^x dx \,.$$

$$L_t^x = \int_0^t \delta(B_s - x)ds\,.$$

$$\begin{aligned} \delta(x) &= \infty \quad \text{if} \quad x = 0 \\ &= 0 \quad \text{if} \quad x \neq 0 \end{aligned}$$

and

$$\int_{\mathbf{R}} \delta(x)dx = 1\,.$$

$$\int_{\mathbf{R}} f(y)\delta(x-y)dy = f(x)\,.$$

is the Dirac delta "function" (generalized function, distribution in the sense of Laurent Schwartz)

Approximate the Dirac delta function by smooth functions

$$\begin{aligned} p_\varepsilon(x) &= \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{|x|^2}{2\varepsilon}} \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} e^{ix\xi - \frac{\varepsilon}{2}\xi^2} d\xi \\ &\to \delta(x) \end{aligned}$$

as $\varepsilon \to 0$.

$$\delta(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{ix\xi} d\xi\,.$$

Self-intersection local time

$$\int_0^t \int_0^t \delta(B_u - B_v) du dv\,.$$

Albeverio, S., Hu, Y. and Zhou, X.Y.

A remark on non smoothness of self-intersection local time of planar Brownian motion.

Statistics and Probability Letter, 32 (1997), 57 - 65.

$\{L_t^x, x \in \mathbf{R}\}$ is a semimartingale

Perkins Edwin

Local time is a semimartingale.

Z. Wahrsch. Verw. Gebiete 60 (1982), no. 1, 79 - 117.

Marcus M. B. and Rosen J.

L_p moduli of continuity of Gaussian processes and local times of symmetric Lévy processes,

Annals of Probab., 36, (2008), 594 - 622.

$$\lim_{h\downarrow 0}\int_{-\infty}^{\infty}\frac{(L_t^{x+h}-L_t^x)^2}{h}dx = 4t\,, \quad \text{a.s.}$$

More generally

$$\begin{aligned} &\lim_{h\downarrow 0}\int_a^b \left|\frac{L_t^{x+h}-L_t^x}{\sqrt{h}}\right|^p dx \\ = \; & 2^p \mathbf{E}\left(|\eta|^p\right)\int_a^b |L_t^x|^{p/2} dx\,, \quad \text{a.s.} \end{aligned}$$

almost surely and also in L^m , $m \geq 1, \eta \sim N(0,1)$.

2. Main results

$$\lim_{h\downarrow 0} \int_{-\infty}^{\infty} \frac{(L_t^{x+h} - L_t^x)^2}{h} dx = 4t\,, \quad \text{a.s.}$$

$$h^{-\frac{1}{2}}\left(\int_R \frac{(L_t^{x+h}-L_t^x)^2}{h}dx-4t\right) \xrightarrow{\mathcal{L}} \frac{8}{\sqrt{3}}\left(\int_R (L_t^x)^2 dx\right)^{\frac{1}{2}}\eta\,,$$

where η is a $N(0,1)$ random variable independent of B and $\mathcal{L}$ denotes the convergence in law.

Chen, X.; Li, W. V.; Marcus, M. B.; Rosen, J.

A CLT for the L^2 modulus of continuity of Brownian local time.

Ann. Probab. 38 (2010), no. 1, 396–438.

Use moment methods

Rosen J.

Derivatives of self-intersection local times.

Preprint

Simpler proof

3. Two tools

Hu, Y. and Nualart, D.

Stochastic integral representation of the L^2 modulus of Brownian local time and a central limit theorem.

Electron. Commun. Probab. 14 (2009), 529–539.

The classical Itô representation theorem asserts that any square integrable random variable can be expressed as

$$F = E[F] + \int_0^\infty u_t dB_t,$$

where $u = \{u_t, t \geq 0\}$ is a unique adapted process such that $E\left(\int_0^\infty u_t^2 dt\right) < \infty$.

If F belongs to $D^{1,2}$, then

$$u_t = E[D_t F|\mathcal{F}_t] \,.$$

This means

$$F = E[F] + \int_0^\infty E[D_t F|\mathcal{F}_t] dB_t.$$

Clark-Ocone formula

One can express the L^2 modulus of local time in terms of the self-intersection local time:

$$G_t(h) = \int_R (L_t(x+h) - L_t(x))^2 dx$$
$$= -2 \int_0^t \int_0^v (\delta(B_v - B_u + h)$$
$$+\delta(B_v - B_u - h)$$
$$-2\delta(B_v - B_u))\, dudv\,.$$

Using Clark-Ocone formula, we have

$$G_t(h) = E(G_t(h)) + \int_0^t u_{t,h}(r) dB_r \, ,$$

where

$$
\begin{aligned}
u_{t,h}(r) &= 4\int_0^r \int_0^h \left(p_{t-r}(B_r - B_u - \eta)\right. \\
&\quad \left. - p_{t-r}(B_r - B_u + \eta)\right) d\eta du \\
&\quad + 4\int_0^r \Big(I_{[0,h]}(B_u - B_r) \\
&\quad - I_{[0,h]}(B_r - B_u)\Big) du.
\end{aligned}
$$

Make the following decomposition

$$u_{t,h}(r) = \hat{u}_{t,h}(r) + 4\Psi_h(r),$$

where

$$\begin{aligned} & \hat{u}_{t,h}(r) = -4 \int_0^r \int_0^h (p_{t-r}(B_r - B_u + \eta) \\ & -p_{t-r}(B_r - B_u - \eta))\, d\eta du \\ = \ & -4 \int_0^r \int_0^h \int_{-\eta}^{\eta} p'_{t-r}(B_r - B_u + \xi) d\xi d\eta du \end{aligned}$$

and

$$\Psi_h(r) = -\int_0^r \big(I_{[0,h]}(B_r - B_u) - I_{[0,h]}(B_u - B_r)\big)\, du.$$

As a consequence

$$G_t(h) - E(G_t(h)) = \int_0^t \hat{u}_{t,h}(r) dB_r + 4 \int_0^t \Psi_h(r) dB_r.$$

The stochastic integral

$$h^{-3/2} \int_0^t \hat{u}_{t,h}(r) dB_r$$

converges in $L^2(\Omega)$ to zero as h tends to zero.

It remains to show the following convergence in law:

$$h^{-\frac{3}{2}} \int_0^t \Psi_h(r) dB_r \xrightarrow{\mathcal{L}} 2\eta\sqrt{\frac{\alpha_t}{3}} \,,$$

$$\begin{aligned} \Psi_h(r) &= -\int_0^r \Big(I_{[0,h]}(B_r - B_u) \\ &\quad - I_{[0,h]}(B_u - B_r) \Big) \, du. \end{aligned}$$

where η is a standard normal random variable independent of B, α_t is given by

$$\begin{aligned}\alpha_t &= \int_R (L_t(x))^2 dx \\ &= \int_0^t \int_0^t \delta(B_s - B_r) dr ds\,.\end{aligned}$$

Notice that

$$M_t^h = h^{-\frac{3}{2}} \int_0^t \Psi_h(r) dB_r$$

is a martingale with quadratic variation

$$\left\langle M^h \right\rangle_t = h^{-3} \int_0^t \Psi_h^2(r)dr.$$

From the asymptotic version of Ray-Knight's theorem it suffices to show the following convergence in probability.

$$h^{-3} \int_0^t \Psi_h^2(r)dr \to \frac{4}{3}\alpha_t,$$

and

$$\left\langle M^h, B \right\rangle_t = h^{-3/2} \int_0^t \Psi_h(r) dr \to 0,$$

as h tends to zero.

Use backward Tanaka formula

Rosen, J.

A CLT for the third integrated moment of Brownian local time.

Preprint.

A central limit theorem for the modulus of continuity in $L^3(R)$ of the local time.

For each fixed $t > 0$

$$\frac{1}{h^2} \int_R (L_t^{x+h} - L_t^x)^3 dx \xrightarrow{\mathcal{L}} 8\sqrt{3} \left(\int_R (L_t^x)^3 dx \right)^{\frac{1}{2}} \eta$$

as h tends to zero, where η is a normal random variable with mean zero and variance one that is independent of B.

4. General case

For each fixed $t > 0$

$$\frac{1}{h^{\frac{p+1}{2}}} \left(\int_R (L_t^{x+h} - L_t^x)^p dx - \int_R E(L_t^{x+h} - L_t^x)^p dx \right) \xrightarrow{\mathcal{L}} \left(\sum_{i=i_0}^{p} C_i \int_R (L_t^x)^i dx \right)^{\frac{1}{2}} \eta$$

as h tends to zero, where η is a normal random variable with mean zero and variance one that is independent of B, and the constants C_i depend on p, t. The index i_0 is 2 if p is even and 3 if p is odd.

Let p be a positive integer. We have the following

$$\begin{aligned} G^h &= \int_{\mathbf{R}} (L_t^{x+h} - L_t^x)^p dx \\ &= E\left[G^h\right] + \sum_{k=1}^{p-1} \int_0^t \Psi_r^{(k)} dB_r , \end{aligned}$$

where for $k = 1, \ldots, p-1$

$$\begin{aligned}\Psi_r^{(k)} &= h^{-\frac{p+1}{2}} \frac{p!}{k!} \int_R \left(L_r^{x+h} - L_r^x\right)^k \\ &\quad \times \left(\int_r^t \left[p'_{s-r}(B_r - x - h) \right.\right. \\ &\quad \left.\left. +(-1)^{p-k} p'_{s-r}(B_r - x) \right] F_{h,k}(s) ds \right) dx,\end{aligned}$$

where, by convention, $F_{h,p-1}(s) = 1$.

$$\begin{aligned} F_{h,k}(s) &= \int_{s<s_{k+2}<\cdots<s_p<t} \prod_{j=k+1}^{p-1} \frac{1}{\sqrt{2\pi(s_{j+1}-s_j)}} \\ &\quad \left(1+(-1)^{p-j} e^{-\frac{h^2}{2(s_{j+1}-s_j)}}\right) ds_{k+2}\cdots ds_p \\ &= \int_{0<\sigma_{p-k-2}+\cdots+\sigma_1<t-s} \prod_{j=1}^{p-k-1} \frac{1}{\sqrt{2\pi\sigma_j}} \\ &\quad \left(1+(-1)^{j} e^{-\frac{h^2}{2\sigma_j}}\right) d\sigma_1\cdots d\sigma_{p-k-2} \end{aligned}$$

$$
\begin{aligned}
F_{h,k}(s) \quad = \quad & h^{\nu} \int_{0<\cdots+\frac{h^2}{u_3}+u_2+\frac{h^2}{u_1}<t-s} \\
& \prod_{j=1, j\mathrm{odd}}^{p-k-1} \frac{1}{\sqrt{2\pi}} u_j^{-\frac{3}{2}} \left(1 - e^{-\frac{u_j}{2}}\right) \\
& \quad \times \prod_{j=1, j\mathrm{even}}^{p-k-1} \frac{1}{\sqrt{2\pi u_j}} \\
& \left(1 + e^{-\frac{h^2}{2u_j}}\right) du_{k+2} \cdots du_p.
\end{aligned}
$$

Thank You