Pruning Galton-Watson Trees and Tree-valued Markov Processes

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Content

Notation for Trees

Introduce the set of labels

$$\mathcal{W} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, \cdots\}$ and by convention $\mathbb{N}^0 = \{\emptyset\}$.

An element of $\mathcal{W}$ is $w = (w^1, \cdots, w^n)$ with $w^i \in \mathbb{N}$.

- If $w = (w^1, \cdots, w^n)$ with $w^i \in \mathbb{N}$, set $|w| = n$ (the generation of w or the height of w).

  $|(1, 2, 1)| = 3$, $|\emptyset| = 0$.

- If $w = (w^1, \cdots, w^m)$ and $v = (v^1, \cdots, v^n)$, write $wv = (w^1, \cdots, w^m, v^1, \cdots, v^n)$ for the concatenation of w and v. ($w\emptyset = \emptyset w = w$).

  $w = (1, 2), v = (1, 3), wv = (1, 2, 1, 3)$.

- ancestors of w:

$$\text{Set } \pi((w^1, \cdots, w^n)) = (w^1, \cdots, w^{n-1}), n \geq 1.$$

$$\pi^k(w) = \pi(\pi^{k-1}(w)), k \leq |w|.$$

$\pi((1, 2, 1)) = (1, 2)$, $\pi((1)) = \emptyset$, $\pi^{|w|}(w) = \emptyset$.

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A (finite or infinite) rooted ordered tree $\mathbf{t}$ is a subset of $\mathcal{W}$ such that

1. $\emptyset \in \mathbf{t}$.
2. $w \in \mathbf{t} \setminus \{\emptyset\} \Longrightarrow \pi(w) \in \mathbf{t}$.
3. For every $w \in \mathbf{t}$, there exists a finite integer $k_w\mathbf{t} \geq 0$ such that, for every $j \in \mathbb{N}$, $wj \in \mathbf{t}$ if and only if $0 \leq j \leq k_w\mathbf{t}$ ($k_w\mathbf{t}$ is the number of children of $w \in \mathbf{t}$).

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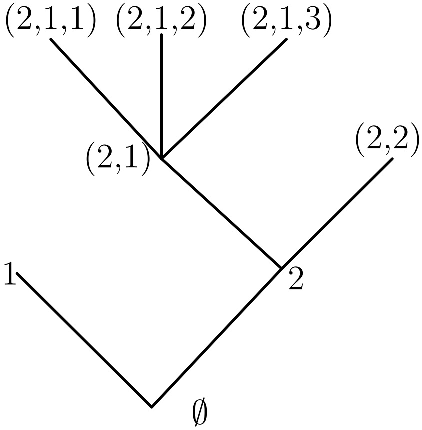
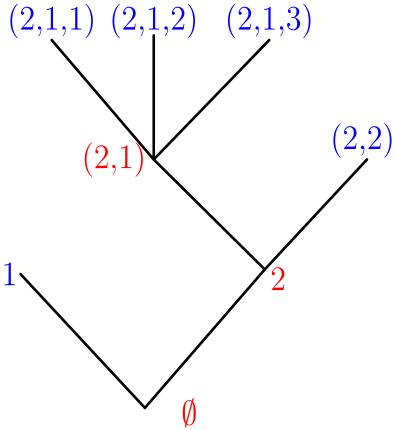


$$|(2,1,3)| = 3$$

$$\pi((2,1,3)) = (2,1)$$

$$\kappa_{(2,1)}\mathbf{t} = 3 \qquad \kappa_{\emptyset}\mathbf{t} = 2$$

For $\nu \in \mathbf{t}$,
if $\kappa_\nu \mathbf{t} > 0$, ν is an inner node.
If $\kappa_\nu \mathbf{t} = 0$, ν is a leaf.



$\emptyset, 2, (2,1)$ are inner nodes

$1, (2,2), (2,1,1), (2,1,2)$
$(2,1,3)$ are leaves

Galton-Watson Trees

Set $\mathcal{N} = \{0, 1, 2, \cdots\}$.
A Galton-Watson tree represents the genealogical structure of the Galton-Watson process:

$$Z_0 = 1, Z_n = \sum_{i=1}^{Z_{n-1}} \eta_i^n, \quad n \geq 1,$$

where $\{\eta_i^n\}$ are i.i.d $\mathcal{N}$-valued random variables. Set

$$p_n = P(\eta_1^1 = n), \quad n \geq 0.$$

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Set

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$$1, \cdots, Z_1$$

$$(1,1), \cdots, (1, \eta_1^1), \cdots, (Z_1, 1), \cdots, (Z_1, \eta_{Z_1}^1)$$

$$\cdots\cdots$$

Let $\mathcal{G}$ denote the set of above labels. Then it is a Galton-Watson tree. For a tree $\mathbf{t}$, we denote by $Z_n\mathbf{t} = \#\{\nu \in \mathbf{t}; |\nu| = n\}$ the number of individuals in the nth generation of $\mathbf{t}$.

$$\begin{cases} \lim_{n\to\infty} Z_n\mathcal{G} = 0, & \sum_n np_n \leq 1((\text{sub})\text{critical}), \\ \lim_{n\to\infty} Z_n\mathcal{G} \in \{0, \infty\}, & \sum_n np_n > 1(\text{supercritical}). \end{cases}$$

$P(\lim_{n\to\infty} Z_n\mathcal{G} = 0)$ is called extinction probability.

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Content

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- **Pruning Trees**
- Tree-valued Processes
- Ascension Process and Its Representation

Marks on Inner Nodes of Trees

Consider a random tree $\mathcal{T}$. Given $\mathcal{T} = \mathbf{t}$, consider independent random variables $(\xi_\nu, \nu \in \mathbf{t}, k_\nu\mathbf{t} \geq 1)$ such that

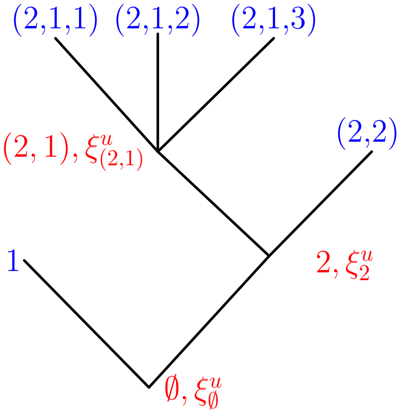
$$P(\xi_\nu \leq u) = u^{k_\nu\mathbf{t}-1}, \quad 0 \leq u \leq 1.$$

Marks on inner nodes:

$$\xi_\nu^u := 1_{\{\xi_\nu \leq u\}}$$

is the mark of an inner node $\nu \in \mathbf{t}$.

Marks on inner nodes: $P(\xi_\nu^u = 1) = u^{k_\nu \mathbf{t} - 1}$



$P(\xi^u_{(2,1)} = 1) = u^2$

$P(\xi^u_2 = 1) = u$

$P(\xi^u_\emptyset = 1) = u$

Pruning Trees

Pruning at node: For $\nu \in \mathcal{T}$ and $0 \leq u \leq 1$, if $\xi_\nu^u = 0$, then all its offsprings (sons, grandsons, $\cdots$) will be removed.

Pruning procedure:

1. We start the pruning procedure from the root.
2. After pruning at nodes in the n-th generation, if there has an un-removed node in $n+1$-th generation, we go on or we stop.

Let us see an example. Denote by $\mathcal{T}(u)$ the set of un-removed nodes in tree $\mathcal{T}$.

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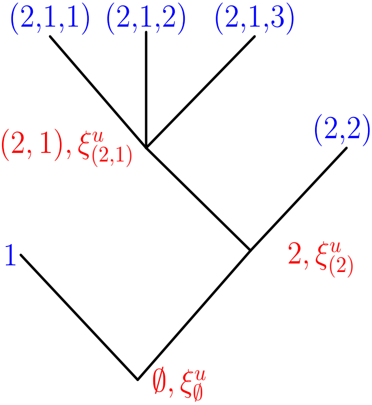
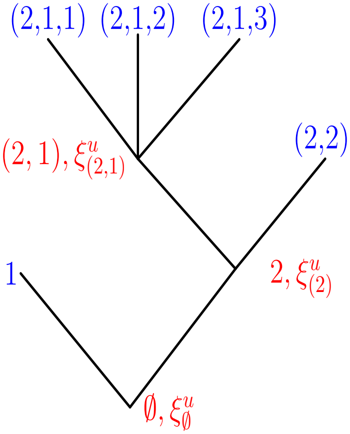
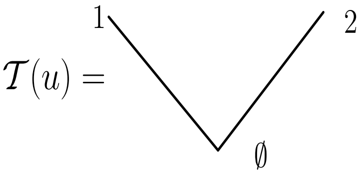
If $\xi^u_\emptyset = 0$,



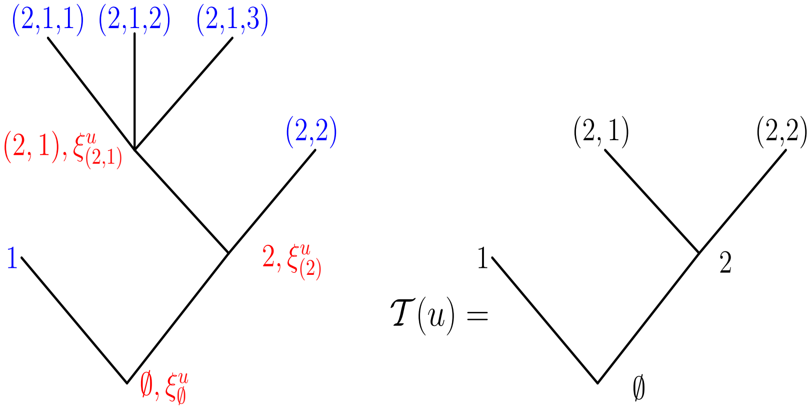
$$\mathcal{T}(u) = \{\emptyset\}$$

If $\xi^u_\emptyset = 1$, $\xi^u_2 = 0$,





If $\xi_{\emptyset}^{u} = 1$, $\xi_{2}^{u} = 1$, $\xi_{(2,1)}^{u} = 0$.



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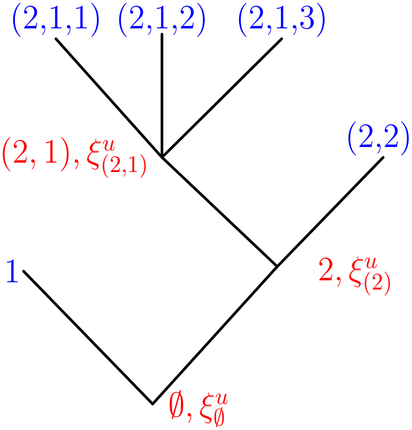

$$\mathcal{T}(u) = \mathcal{T}$$

Pruning Galton-Waston trees

Proposition

If $\mathcal{T}$ is a Galton Watson tree with offspring distribution $\{p_n, n \geq 0\}$, then $\mathcal{T}(u)$ is also a Galton Watson tree with offspring distribution $\{p_n^{(u)}, n \geq 0\}$ defined by

$$\begin{cases} p_n^{(u)} = u^{n-1} p_n, & n \geq 1, \\ p_0^{(u)} = 1 - \sum_{n\geq 1} u^{n-1} p_n & . \end{cases}$$

Content

Suppose $\mathcal{T}$ is a Galton-Watson tree with offspring distribution $\{p_n, n \geq 0\}$. For every $u \in [0, 1]$, set

$$\mathcal{T}(u) = \left\{\nu \in \mathcal{T},\ \forall 1 \leq n \leq |\nu|,\ \xi_{\pi^n(\nu)} \leq u\right\}.$$

We get a stochastic process $\{\mathcal{T}(u) : 0 \leq u \leq 1\}$ such that

1. $\mathcal{T}(\alpha) \subset \mathcal{T}(\beta)$ for $\alpha \leq \beta$.
2. $\mathcal{T}(u)$ is a Galton-Watson tree with distribution $\{p_n^{(u)}, n \geq 0\}$.
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Note that

$$p_n^{(\alpha)} = (\alpha/\beta)^{n-1} p_n^{(\beta)}, \quad n \geq 1.$$

Let $\hat{\mathcal{T}}(\alpha)$ be a random tree obtained by pruning $\mathcal{G}(\beta)$ with parameter α/β.
Then

$$(\hat{\mathcal{T}}(\alpha), \mathcal{T}(\beta)) \stackrel{d}{=} (\mathcal{T}(\alpha), \mathcal{T}(\beta)).$$

Question

How to obtain $\mathcal{T}(\beta)$ given $\mathcal{T}(\alpha)$?

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Modified Galton-Watson trees

Recall

$$\begin{cases} p_n^{(u)} = u^{n-1} p_n, & n \geq 1, \\ p_0^{(u)} = 1 - \sum_{n\geq 1} u^{n-1} p_n & . \end{cases}$$

Fix α and β with $0 \leq \alpha \leq \beta \leq 1$. Define

$$\begin{cases} p_{\alpha,\beta}(k) = \frac{(1-(\alpha/\beta)^{k-1}) p_k^{(\beta)}}{p_0^{(\alpha)}}, & \text{for } k \geq 1, \\ p_{\alpha,\beta}(0) = \frac{p_0^{(\beta)}}{p_0^{(\alpha)}} & . \end{cases}$$

$\{p_{\alpha,\beta}(n); n \geq 0\}$ is a probability distribution.

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Define a random tree $\mathcal{T}_{\alpha,\beta}$ such that

$$\mathcal{P}(\mathcal{T}_{\alpha,\beta} = \mathbf{t}) = \mathcal{P}(\mathcal{T}(\beta) = \mathbf{t} \mid k_\emptyset \mathcal{T}(\beta) = k_\emptyset \mathbf{t}) p_{\alpha,\beta}(k_\emptyset \mathbf{t}). \tag{1}$$

This means that $\mathcal{T}_{\alpha,\beta}$ is a modified Galton Watson tree, in which

1. the size of the first generation has distribution $p_{\alpha,\beta}$;
2. all subsequent individuals have offspring distribution $p^{(\beta)}(\cdot)$.

Growth of Galton-Watson trees

Let $\mathcal{L}(\alpha)$ be the set of leaves of $\mathcal{T}(\alpha)$. Given $\mathcal{T}(\alpha)$, let $(\mathcal{T}^{\nu}_{\alpha,\beta},\ \nu \in \mathcal{L}(\alpha))$ be i.i.d. random trees with distribution $\mathcal{T}_{\alpha,\beta}$.
Set

$$\hat{\mathcal{T}}(\beta) = \mathcal{T}(\alpha) \cup \bigcup_{\nu \in \mathcal{L}(\alpha)} \{\nu w : w \in \mathcal{T}^{\nu}_{\alpha,\beta}\}. \tag{2}$$

$\hat{\mathcal{T}}(\beta)$ is a random tree obtained by adding a modified Galton Watson tree $\mathcal{T}^{\nu}_{\alpha,\beta}$ on each leaf ν of $\mathcal{T}(\alpha)$.

Proposition

$$(\mathcal{T}(\alpha), \mathcal{T}(\beta)) \stackrel{d}{=} (\mathcal{T}(\alpha), \hat{\mathcal{T}}(\beta)).$$

A Galton-Watson Tree Conditioned on Non-Extinction

Let $\{p_n, n \geq 0\}$ be a critical or subcritical offspring distribution with $p_0 < 1$, i.e.

$$\sum_{n\geq 1} np_n \leq 1.$$

Let $\mathcal{T}$ be a Galton Watson tree with offspring distribution p. Then $\mathcal{T}$ is finite almost surely. ($\lim_{n\to\infty} Z_n\mathcal{T} = 0, a.s.$)

We first recall a result in Kesten (1987) and Aldous and Pitman(1998).

1. There exists a random tree $\mathcal{T}^\infty$ such that

$$dist(\mathcal{T} | \{Z_n \mathcal{T} > 0\}) \rightarrow \mathcal{T}^\infty$$

2. Almost surely $\mathcal{T}^\infty$ contains a unique infinite path.

- We could construct a tree-valued process $\{\mathcal{T}^*(u); 0 \leq u \leq 1\}$ by pruning $\mathcal{T}^\infty$.
- For each $u < 1$, $\mathcal{T}^*(u)$ is almost surely finite whose distribution could be represented in terms of $\mathcal{T}(u)$.

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We first recall a result in Kesten (1987) and Aldous and Pitman(1998).

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Content

Assumptions on Offspring Distribution

Suppose $\{p_n : n = 0, 1, \cdots\}$ is a probability distribution with $p_1 < 1$ and $\sum_n np_n = 1$. Assume that

$$\textit{there exists a constant } \bar{u} > 1 \textit{ such that } \sum_{n=1}^{\infty} \bar{u}^{n-1} p_n = 1.$$

For $u \in [0, \bar{u}]$, define

$$\begin{cases} P_u(k) = u^{k-1} p_k, & k \geq 1, \\ P_u(0) = 1 - \sum_{k=1}^{\infty} P_u(k). \end{cases} \tag{3}$$

Ascension Process

Using the pruning at nodes procedure, we construct a tree-valued process $(\mathcal{G}(u), 0 \leq u \leq \bar{u})$ such that

- The process $(\mathcal{G}(t\bar{u}),\ t \in [0, 1])$ is obtained by pruning $\mathcal{G}(\bar{u})$,
- for every u, $\mathcal{G}(u)$ is a Galton-Watson tree with offspring distribution $P_u(\cdot)$,
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We now consider $\{\mathcal{G}(u), 0 \leq u \leq \bar{u}\}$ as an *ascension process* with the *ascension time*

$$A := \inf\{u \in [0, \bar{u}],\ \mathcal{G}(u) \text{ is infinite. }\}$$

with the convention $\inf \emptyset = \bar{u}$.

Denote by $\mathcal{G}^{\infty}(1)$ the infinite tree by conditioning $\mathcal{G}(1)$ to be non-extinction.

Let $\{\mathcal{G}^{*}(u); 0 \leq u \leq 1\}$ be the tree-valued process obtained by pruning $\mathcal{G}^{\infty}(1)$.

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Representation of the Ascension Process

Denote by $F(u)$ the extinction probability of a Galton Watson process with offspring distribution $P_u(\cdot)$.

Proposition

$$\{\mathcal{G}(u),\ 0 \leq u < A\} \stackrel{d}{=} \{\mathcal{G}^*(u\gamma) : 0 \leq u < \bar{F}^{-1}(1-\gamma)\},$$

where $\bar{F}^{-1} : [0,1] \to [1,\bar{u}]$ *is the inverse function of* $\bar{F}$ *and* γ *is a r.v. uniformly distributed on* $(0,1)$, *independent of* $\{\mathcal{G}^*(u) : 0 \leq u \leq 1\}$.

Some Remarks

1. Aldous and Pitman (1998) obtained similar results for Galton-Watson trees by uniform pruning (adding marks on edges). The offspring distribution there is a poisson distribution.
2. For pruning Levy trees and continuum tree valued processes, see Abraham, Delmas and Voisin (2010) and Abraham, Delmas (2010).
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Thanks!