Large deviations under sublinear expectations

Fuqing Gao

Wuhan University

The 7th Workshop Markov Processes and Related Topics
Beijing Normal University, July 19-23, 2010

Outline

Sublinear expectations

First, let us briefly recall some basic conceptions and results about sublinear-expectations (see [11] for details).

- $\mathbb{S}(d)$: the collection of $d \times d$ symmetric matrices. $\langle R, Q\rangle = tr[RQ]$ for any $R, Q \in \mathbb{S}(d)$. $\mathbb{S}_+(d)$: the set of the nonnegative elements in $\mathbb{S}(d)$.
- $\mathcal{H}$: a linear space of real functions defined on a Polish space Ω such that $C_b(\Omega) \subset \mathcal{H}$ and if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{b,Lip}(\mathbb{R}^n)$.
- Sublinear expectation: $\mathbb{E}[\cdot] : X \in \mathcal{H} \mapsto \mathbb{E}(X) \in \mathbb{R}$:
  - (a) Monotonicity: If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
  - (b) Constant preserving: $\mathbb{E}[c] = c$, for all $c \in \mathbb{R}$;
  - (c) Sub-additivity: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$;
  - (d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for all $\lambda \geq 0$.
- The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H})$.

Independence and identical distribution

- A m-dimensional random vector $X = (X_1, \cdots, X_m)$ is said to be independent of another n-dimensional random vector $Y = (Y_1, \cdots, Y_n)$ if

$$\mathbb{E}(\varphi(X, Y)) = \mathbb{E}(\mathbb{E}(\varphi(X, y))_{y=Y}), \quad \text{for } \varphi \in C_{b,Lip}(\mathbb{R}^m \times \mathbb{R}^n).$$

- Let X_1 and X_2 be two d–dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}$ They are called identically distributed, denoted by $X_1 \sim X_2$, if

$$\mathbb{E}_1(\varphi(X_1)) = \mathbb{E}_2(\varphi(X_2)), \quad \forall \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

- A sequence of d-dimensional random variables $\{X_n, n \geq 1\}$ with each component being in the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is said to be i.i.d, if each X_{i+1} is independent of $(X_1, \cdots, X_i)$, and $X_{i+1} \sim X_i$ for $i = 1, 2, \cdots$.

- Throughout this talk, we only consider regular sublinear expectation, i. e. , for all $\{X_n, n \in \mathbb{N}\} \subset \mathcal{H}$, $X_n(\omega) \downarrow 0$ for all $\omega \in \Omega \implies \lim_{n\to\infty} \mathbb{E}(X_n) = 0$.
- Under the regular condition, there exists a relatively compact subset $\mathcal{P}$ of $\mathcal{M}$ (the space of probability measures on Ω) such that $\mathbb{E}(X) = \sup_{P\in\mathcal{P}} E_P(X) := \overline{\mathbb{E}}(X)$ for each $X \in L^1_G(\Omega)$.
- The natural Choquet capacity associated with $\mathbb{E}$ is defined by

$$c(A) := \sup_{P\in\mathcal{P}} P(A).$$

- A map $G : \mathbb{S}(d) \mapsto \mathbb{R}$ is said to be a monotonic and sublinear function, if for $A, \bar{A} \in \mathbb{S}(d)$,

$$\begin{cases} G(A+\bar{A}) \leq G(A) + G(\bar{A}), \\ G(\lambda A) = \lambda G(A), \quad \text{for all } \lambda \geq 0, \\ G(A) \geq G(\bar{A}), \quad \text{if } A \geq \bar{A}. \end{cases} \tag{1.1}$$

- Given a monotonic and sublinear function $G : \mathbb{S}(d) \mapsto \mathbb{R}$, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_+(d)$ such that

$$G(A) = \frac{1}{2} \sup_{\sigma \in \Sigma} (A, \sigma).$$

We assume that there exist constants $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$ such that

$$\Sigma \subset \left\{ \sigma \in \mathbb{S}_d;\ \underline{\sigma} I_{d\times d} \leq \sigma \leq \overline{\sigma} I_{d\times d} \right\}. \tag{1.2}$$

G-normal distribution

- A d-dimensional random vector $X = (X_1, \cdots, X_d)$ is called **G-normal distributed** if for each $\varphi \in lip(\mathbb{R}^d)$,

$$u(t,x) := \mathbb{E}\left(\varphi(x+\sqrt{t}X)\right), \quad t \geq 0, \; x \in \mathbb{R}^d$$

is the unique viscosity solution of the following nonlinear heat kernel equation:

$$\frac{\partial u}{\partial t} = G\left(D_x^2 u\right), \quad t \geq 0, \; x \in \mathbb{R}^d; \quad u(0,x) = \varphi(x), \tag{1.3}$$

where $D_x^2 u = (\partial^2_{x_i x_j} u)_{i,j=1}^d$ is the Hessian matrix of u.

- A d-dimensional random vector η is called **maximal distributed** if there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d$ such that for any $\varphi \in C_{b,Lip}$,

$$\mathbb{E}[\varphi(\eta)] = \max_{y \in \Gamma} \varphi(y).$$

LLNs and CLT

- Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. $\mathbb{R}^d$-valued random variables on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Set $\underline{\mu} = -\mathbb{E}(-X_1)$, $\overline{\mu} = \mathbb{E}(X_1)$, and $\{\bar{S}_n := \frac{1}{n}\sum_{i=1}^n X_i, n \geq 1\}$
- (LLN, Peng ([9])): For any $\varphi \in C_{b,Lip}$, $\lim_{n\to\infty} \mathbb{E}(\varphi(\bar{S}_n)) = \mathbb{E}(\varphi(\eta))$.
- (CLT, Peng ([9])): Assume that $\underline{\mu} = \overline{\mu} = 0$. Then

$$\lim_{n\to\infty} \mathbb{E}(\varphi(\sqrt{n}\bar{S}_n)) = \mathbb{E}(\varphi(X)),$$

  where X is a G-normal distributed random vector and $G(A) = \frac{1}{2}\mathbb{E}(\langle AX_1, X_1\rangle)$.
- (SLLN, Chen([2]), Maccheroni and Marinacci([7])): For $d = 1$,

$$c\left(\{\underline{\mu} > \liminf_{n\to\infty} \bar{S}_n\} \cup \{\limsup_{n\to\infty} \bar{S}_n > \overline{\mu}\}\right) = 0.$$

G-Brownian motion

- Let Ω denote the space of all $\mathbb{R}^d$-valued continuous paths $\omega : (0, +\infty)$ $t \longmapsto \omega_t \in \mathbb{R}^d$, with $\omega_0 = 0$.
- For each $t > 0$, set

$$L_{ip}(\Omega_t) := \left\{ \varphi\left(\omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}\right) : n \geq 1, t_1, \cdots, t_n \in [0, t], \varphi \in lip(\mathbb{R}^{d\times n}) \right.$$

- A continuous process $\{B_t(\omega)\}_{t\geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E}^G)$ is called a ***G*-Brownian motion** if the following properties are satisfied:
  (i). $B_0 = 0$, B_1 is G-normal distributed and $\mathbb{E}^G(B_t) = -\mathbb{E}^G(-B_t) = 0$ for $t \geq 0$.
  (ii). For any $s, t \geq 0$, $B_{t+s} - B_s \sim B_t$.
  (iii). For any $m \geq 1$, $0 = t_0 < t_1 < \cdots < t_m < \infty$, the increment $B_{t_m} - B_{t_{m-1}}$ is independent from $B_{t_1}, \cdots, B_{t_{m-1}}$.

The representation theorem of G-expectation

- The topological completion of $L_{ip}(\Omega_t)$ (resp. $L_{ip}(\Omega)$) under the Banach norm $\|\cdot\|_{p,G} := (\mathbb{E}^G(|\cdot|^p))^{1/p}$ is denoted by $L_G^p(\Omega_t)$ (resp. $L_G^p(\Omega)$), where $p \geq 1$. $\mathbb{E}^G(\cdot)$ can be extended uniquely to a sublinear expectation on $L_G^1(\Omega)$. We denote also by $\mathbb{E}^G$ the extension.
- Set $\Gamma := \{\gamma = \sigma^{1/2}, \sigma \in \Sigma\}$. Let P be the Wiener measure on Ω. Let $\mathcal{A}_{0,\infty}^{\Gamma}$ be the collection of all Γ-valued $\{\mathcal{F}_t, t \geq 0\}$-adapted processes on the interval $[0, +\infty)$, and let P_θ be the law of the process $\{\int_0^t \theta_s d\omega_s, t \geq 0\}$ under the Wiener measure P. Then ([4]): under $\sup_{\theta \in \mathcal{A}_{0,\infty}^{\Gamma}} E_{P_\theta}$, the canonical process B is G-Brownian motion, and for all $X \in L_G^1(\mathcal{F})$

$$\mathbb{E}^G(X) = \sup_{\theta \in \mathcal{A}_{0,\infty}^{\Gamma}} E_{P_\theta}(X). \tag{1.4}$$

G-stochastic integral

For $p \in [1,\infty)$, let $M_G^{p,0}(0,1)$ denote the space of $\mathbb{R}$-valued piecewise constant processes

$$H = \sum_{i=0}^{n-1} H_{t_i} 1_{[t_i, t_{i+1})}$$

where $H_{t_i} \in L_G^p(\Omega_{t_i})$, $0 = t_0 < t_1 < \cdots < t_n = 1$. For $H \in M_G^{p,0}(0,1)$, $j = 1, \cdots, d$, the G-stochastic integral is defined by

$$I^j(H) := \int_0^t H_s dB_s^j := \sum_{i=0}^{n-1} H_{t_i}(B^j_{t\wedge t_{i+1}} - B^j_{t\wedge t_i}).$$

Let $M_G^p(0,1)$ be the closure of $M_G^{p,0}(0,1)$ under the norm:

$$\|H\|^p_{M_G^p(0,1)} := \mathbb{E}^G\left(\int_0^1 |H_t|^p dt\right).$$

Then the mapping $I^j : M_G^{2,0}(0,1) \to L_G^2(\Omega_1)$ is continuous, and so it can be continuously extended to $M_G^2(0,1)$.

Quadratic variation process

The quadratic variation process of G-Brownian motion is defined by

$$\langle B\rangle_t := (\langle B\rangle_t^{ij})_{1\le i,j\le d} = \left(B_t^i B_t^j - 2\int_0^t B_s^i dB_s^j\right)_{1\le i,j\le d}, \qquad t\ge 0.$$

For $H\in (M_G^1(0,1))^d$, define

$$\int_0^t H_s d\left\langle B\right\rangle_s = \left(\sum_{j=1}^d \int_0^t H_s^1 d\left\langle B\right\rangle_s^{j1}, \cdots, \sum_{j=1}^d \int_0^t H_s^d d\left\langle B\right\rangle_s^{jd}\right)^T.$$

and for $H\in (M_G^1(0,1))^{d\times d}$, define

$$\int_0^t H_s d\left\langle B\right\rangle_s = \sum_{i,j=1}^d \int_0^t H_s^{ij} d\left\langle B\right\rangle_s^{ij}.$$

LDP for independent random variables

Large deviations and moderate deviations for i.i.d. random variables are based on joint work with Mingzhou Xu.

Theorem 2.1

Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. $\mathbb{R}^d$-valued random variables. Assume that there is a $\delta > 0$, such that $\bar{\mathbb{E}}[e^{\delta|X_1|}] < \infty$. Then there exists a rate function $I : \mathbb{R}^d \mapsto [0, \infty]$ such that for any open set $O \in \mathcal{B}(\mathbb{R}^d)$,

$$\liminf_{N\to\infty} \frac{1}{N} \log c\left(\frac{1}{N}\sum_{i=1}^{N} X_i \in O\right) \geq -\inf_{x\in O} I(x),$$

and for any closed subset $F \in \mathcal{B}(\mathbb{R}^d)$,

$$\limsup_{N\to\infty} \frac{1}{N} \log c\left(\frac{1}{N}\sum_{i=1}^{N} X_i \in F\right) \leq -\inf_{x\in F} I(x).$$

- If $\bar{\mathbb{E}}[e^{\delta|X_1|}] < \infty$ for all $\delta > 0$, then

$$I(x) = \sup_{\alpha \in \mathbb{R}^d} \left\{ \langle x, \alpha \rangle - \log \bar{\mathbb{E}} \left(e^{\langle \alpha, X_1 \rangle} \right) \right\}.$$

- The large deviation principle is established by the subadditive method. The representation of the rate function is obtain by the Varadhan asymptotical integral lemma under the capacity.

- If $\mathcal{P}$ is a convex and compact set, then

$$I(x) = \inf_{P\in\mathcal{P}} \sup_{\alpha\in\mathbb{R}^d} \left\{ \langle x, \alpha \rangle - \log E_P \left(\mathrm{e}^{\langle \alpha, X_1 \rangle} \right) \right\}.$$

- For any $P \in \mathcal{P}$, $I(E_P(X_1)) = 0$, i.e., the solutions of the equation $I(x) = 0$ are not unique under uncertainty of mean.
- If $X_1 \sim N(0, \Sigma)$, where Σ is a compact convex subset and for some $0 < \underline{\sigma} \le \bar{\sigma} < \infty$,

$$\Sigma \subset \{\sigma \in \mathbb{S}_+(d); \underline{\sigma} I_{d\times d} \le \sigma \le \bar{\sigma} I_{d\times d}\},$$

then

$$I(x) = \frac{1}{2} \inf_{\sigma\in\Sigma} \langle x, \sigma x \rangle.$$

Large deviations for the empirical measures

Theorem 2.2

For any closed set $F \in \mathcal{B}\left(M_1(\mathbb{R}^d)\right)$, *we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log c\left(\frac{1}{n}\left(\delta_{X_1}(B) + \cdots + \delta_{X_n}(B)\right) \in F\right) \leq -\inf_{\nu\in F} I_c(\nu),$$

and for any open set $G \in \mathcal{B}\left(M_1(\mathbb{R}^d)\right)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log c\left(\frac{1}{n}\left(\delta_{X_1}(B) + \cdots + \delta_{X_n}(B)\right) \in G\right) \geq -\inf_{\nu\in G} I_c(\nu),$$

where

$$I_c(\nu) := \sup_{f\in C_b(\mathbb{R}^d)} \left\{\langle f, \nu\rangle - \log \bar{\mathbb{E}}\left(\exp(f(X_1))\right)\right\}.$$

- Define the relative entropy a probability ν w.r.t c:

$$Ent_c(\nu) = \inf_{P \in \mathcal{P}} h(\nu, \mu_P)$$

where $\mu_P = P \circ X_1^{-1}$ and

$$h(\nu, \mu_P) = \begin{cases} \int_{\mathbb{R}^d} \left(\frac{d\nu}{d\mu_P} \log \frac{d\nu}{\mu_P} \right) d\mu_P & \text{if } \nu \ll \mu_P, \\ +\infty & \text{otherwise.} \end{cases}$$

- In addition, if $\mathcal{P}$ is a convex and compact set, then

$$I_c(\nu) = Ent_c(\nu),$$

- For any $P \in \mathcal{P}$, $I_c(\mu_P)) = 0$, i.e., the solutions of the equation $I_c(\nu) = 0$ are not unique under uncertainty of mean.

MDP for independent random variables

- Let $\{(X_n, Y_n), n \geq 1\}$ be i.i.i. $\mathbb{R}^d \times \mathbb{R}^d$-valued random variables in $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that
  (*i*). $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$;
  (*ii*). there is a $\delta \in (0,1)$, such that $\bar{\mathbb{E}}[|X_1|^{2+\delta}] < \infty$ and $\bar{\mathbb{E}}[|Y_1|^2] < \infty$.
- Let $\{a(n); n \geq 1\}$ be a sequence of positive real numbers satisfying

$$\frac{n}{a(n)} \uparrow \infty, \quad \frac{a(n)}{n^{1/2}} \uparrow \infty, \text{ as } n \to \infty.$$

-

$$J(x) := \sup_{\alpha \in \mathbb{R}^d} \left\{ \langle x, \alpha \rangle - \mathbb{E}\left(\langle \alpha, Y_1 \rangle + \frac{1}{2} \langle \alpha \alpha^\tau X_1, X_1 \rangle \right) \right\}. \tag{2.1}$$

- If $\mathcal{P}$ is a convex and compact set, then

$$J(x) = \inf_{P \in \mathcal{P}} \frac{1}{2} \langle (x - E_P(Y_1)), (E_P(X_1 X_1^T))^{-1} (x - E_P(Y_1)) \rangle.$$

Theorem 2.3

For any open subset $O \in \mathcal{B}(\mathbb{R}^d)$,

$$\liminf_{n\to\infty} \frac{n}{a^2(n)} \log c\left(\frac{1}{a(n)}\sum_{i=1}^{n} X_i + \frac{1}{n}\sum_{i=1}^{n} Y_i \in O\right) \geq -\inf_{x\in O} J(x). \quad (2.2)$$

Theorem 2.4

Assume that

$$\lim_{n\to\infty} \frac{n}{a^2(n)} \log\left(nc\left(\max\{|X_1|, |Y_1|\} \geq a(n)\right)\right) = -\infty. \quad (2.3)$$

Then, for any closed subset F *in* $\mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{n}{a^2(n)} \log c\left(\frac{1}{a(n)}\sum_{i=1}^{n} X_i + \frac{1}{n}\sum_{i=1}^{n} Y_i \in F\right) \leq -\inf_{x\in F} J(x). \quad (2.4)$$

Peng's CLT and the large deviations for G-distributed random variables play important role in the proof of lower bound. The upper bound is proved by the truncation technique and the Laplace asymptotic integral method.

Variational representation

Theorem 3.1

Let $\Phi \in L_G^1(\Omega_1)$ be bounded. Then

$$\mathbb{E}^G(e^{\Phi(B.)})$$
$$= \exp\left\{ \sup_{H \in (M_G^2(0,1))^d} \mathbb{E}^G\left(\Phi\left(B. + \int_0^{\cdot} H_s d\langle B\rangle_s \right) - \frac{1}{2}\int_0^1 H_s H_s^T d\langle B\rangle_s \right) \right\}.$$

- In the classical case, a variational representation of functionals of finite dimensional Brownian motion was obtained by Boué and Dupuis ([1]).
- Under the G-expectation, the complicated measurable selection technique in [1] and the Clark-Ocone formula cannot be used. We use a new approach to overcome these difficulties. The key of the upper bound is to define an appropriate function $H \in (M_G^2(0,1))^d$ using a sequence of stochastic differential equations.

Variational representation

Theorem 3.1

Let $\Phi \in L^1_G(\Omega_1)$ *be bounded. Then*

$$\mathbb{E}^G(e^{\Phi(B_\cdot)})$$
$$= \exp\left\{ \sup_{H \in (M^2_G(0,1))^d} \mathbb{E}^G\left(\Phi\left(B_\cdot + \int_0^\cdot H_s d\langle B\rangle_s \right) - \frac{1}{2}\int_0^1 H_s H_s^T d\langle B\rangle_s \right) \right\}.$$

- In the classical case, a variational representation of functionals of finite dimensional Brownian motion was obtained by Boué and Dupuis ([1]).
- Under the G-expectation, the complicated measurable selection technique in [1] and the Clark-Ocone formula cannot be used. We use a new approach to overcome these difficulties. The key of the upper bound is to define an appropriate function $H \in (M^2_G(0,1))^d$ using a sequence of stochastic differential equations.

Theorem 3.2

For any closed subset F in $\left(C\left([0,1],\mathbb{R}^p\right),\rho\right)$,

$$\limsup_{\varepsilon\to 0}\varepsilon\log c^G\left(\sqrt{\varepsilon}B|_{[0,1]}\in F\right)\leq-\inf_{\psi\in F}J(h) \quad (3.1)$$

and for any open subset O in $\left(C\left([0,1],\mathbb{R}^p\right),\rho\right)$,

$$\liminf_{\varepsilon\to 0}\varepsilon\log c^G\left(\sqrt{\varepsilon}B|_{[0,1]}\in O\right)\geq-\inf_{\psi\in O}J(h), \quad (3.2)$$

where

$$J(h)=\begin{cases}\frac{1}{2}\int_0^1\inf_{\sigma\in\Sigma}(h'(s),\sigma^{-1}h'(s))ds, & h\in\mathbb{H},\\ +\infty, & \textit{otherwise},\end{cases} \quad (3.3)$$

and $\mathbb{H}=\left\{f(\cdot)=\int_0^\cdot f'(s)ds;\, f'\in L^2([0,1],\mathbb{R}^d)\right\}$.

LDP for stochastic flows

Consider the following small perturbation stochastic differential equation driven by d-dimensional G-Brownian motion B:

$$X^\varepsilon(x,t) = x + \int_0^t b^\varepsilon(X^\varepsilon(x,s))ds + \sqrt{\varepsilon}\int_0^t \sigma^\varepsilon(X^\varepsilon(x,s))dB_s, \qquad (3.4)$$

where

$$b^\varepsilon = (b_1^\varepsilon, \cdots, b_p^\varepsilon)^T : \mathbb{R}^p \to \mathbb{R}^p; \quad \sigma^\varepsilon = (\sigma_{i,j}^\varepsilon) : \mathbb{R}^p \to \mathbb{R}^p \otimes \mathbb{R}^d, \ \varepsilon \geq 0$$

satisfy the following conditions:

$(H1)$. b^ε and σ^ε, $\varepsilon \geq 0$ are uniformly Lipschitz continuous.

$(H2)$. b^ε and σ^ε converge uniformly to $b := b^0$ and $\sigma := \sigma^0$, respectively.

For positive number $p \geq 1$ given, for each $N \geq 1$, $\psi \in C(\mathbb{R}^p \times [0,1], \mathbb{R}^p)$, set

$$\|\psi\|_N = \sup_{|x| \leq N, t \in [0,1]} |\psi(x,t)|,$$

and define

$$\rho(\psi_1, \psi_2) = \sum_{N=1}^{\infty} \frac{1}{2^N} \min\{\|\psi_1 - \psi_2\|_N, 1\}, \quad \psi_1, \psi_2 \in C(\mathbb{R}^p \times [0,1], \mathbb{R}^p).$$

Then $(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p), \rho)$ is a separable metric space.
For any $f \in \mathbb{H}$, let $\Psi(f)(x,t) \in C(\mathbb{R}^p \times [0,1], \mathbb{R}^p)$ be a unique solution of the following ordinary differential equation:

$$\Psi(f)(x,t) = x + \int_0^t b(\Psi(f)(x,s))ds + \int_0^t \sigma(\Psi(f)(x,s))f'(s)ds \qquad (3.5)$$

$$I(\psi) = \inf_{h \in \mathbb{H}} \{J(h),\ \psi = \Psi(h)\}, \quad \psi \in C(\mathbb{R}^p \times [0,1], \mathbb{R}^p). \qquad (3.6)$$

Theorem 3.3

Let (H1) and (H2) hold. Let $X^\varepsilon = \{X^\varepsilon(x,t), x \in \mathbb{R}^p, t \in [0,1]\}$ be a unique solution of the SDE (3.4). Then for any closed subset F in $(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p), \rho)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log c^G (X^\varepsilon \in F) \leq -\inf_{\psi\in F} I(\psi) \tag{3.7}$$

and for any open subset O in $(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p), \rho)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log c^G (X^\varepsilon \in O) \geq -\inf_{\psi\in O} I(\psi), \tag{3.8}$$

From the variational representation, we can prove that for any $\Phi \in C_b(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p))$,

$$\lim_{\varepsilon\to 0} \left| \varepsilon \log \mathbb{E}^G \left(\exp\left\{ \frac{\Phi(X^\varepsilon)}{\varepsilon} \right\} \right) - \sup_{\psi\in C(\mathbb{R}^p\times[0,1],\mathbb{R}^p)} \{\Phi(\psi) - I(\psi)\} \right| = 0. \tag{3.9}$$

Theorem 3.3

Let (H1) and (H2) hold. Let $X^\varepsilon = \{X^\varepsilon(x,t), x \in \mathbb{R}^p, t \in [0,1]\}$ *be a unique solution of the SDE (3.4). Then for any closed subset F in* $(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p), \rho)$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log c^G(X^\varepsilon \in F) \leq -\inf_{\psi \in F} I(\psi) \tag{3.7}$$

and for any open subset O in $(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p), \rho)$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log c^G(X^\varepsilon \in O) \geq -\inf_{\psi \in O} I(\psi), \tag{3.8}$$

From the variational representation, we can prove that for any $\Phi \in C_b(C(\mathbb{R}^p \times [0,1], \mathbb{R}^p))$,

$$\lim_{\varepsilon \to 0} \left| \varepsilon \log \mathbb{E}^G \left(\exp\left\{ \frac{\Phi(X^\varepsilon)}{\varepsilon} \right\} \right) - \sup_{\psi \in C(\mathbb{R}^p \times [0,1], \mathbb{R}^p)} \{\Phi(\psi) - I(\psi)\} \right| = 0. \tag{3.9}$$

- Boué, M. and Dupuis, P., A variational representation for certain functionals of Brownian motion. *Ann. Probab.*, 26(1998), 1641-1659.
- Chen Z. J., Strong laws of numbers for capacities. arXiv:1006:0749v1,2010
- Dembo, J. and Zeitouni, O., *Large deviations Techniques and Applications*. Springer, New York, 2nd edition, 1998.
- Denis, L., Hu, M. S. and Peng, S., Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes. arXiv: math.PR/0802.1240, 2008.
- Gao, F. Q., Pathwise properties and homeomorphic flows for stochastic differential equatiosn driven by G-Brownian motion. *Stoch. Proc. Appl.*, 119(2009), 3356-3382.
- Gao, F. Q. and Jiang, H., Large Deviations for Stochastic Differential Equations Driven by G-Brownian Motion. *Stoch. Proc. Appl.*, to appear.

- Maccheroni, F. and Marinacci, M.: A strong law of large numbers for capacities. *Ann. Probab.* 33(2005), 1171-1178.
- Peng, S. G.: G-Expectation, G-Brownian motion and related stochastic calculus of Itô's type, in: *Proceedings of the 2005 Abel Symposium 2,* Edit. Benth et. al. 541–567, Springe-Verlag, 2006.
- Peng, S. G: A New Central Limit Theorem under Sublinear
- Peng, S. G., Multi-dimensional G-Brownian Motion and related stochastic calculus under G-expectation. *Stoch. Proc. Appl.*, 118(2008), 2223–2253.
- Peng, S. G, Nonlinear Expectations and Stochastic Calculus under Uncertainty. arXiv:math.PR/1002.4546, 2010.
- Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*. Grund. Math. Wiss. 293, Springer-Verlag, 1998.
- Soner, H. M., Touzi, N. and Zhang J. F., Martingale representation theorem for the G-expectation. arXiv:math.PR/1001.3802, 2010.

Thank You