Consistent Minimal Displacement of Branching Random Walks

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(Independent Work from G. Faraud, Y. Hu and Z. Shi)

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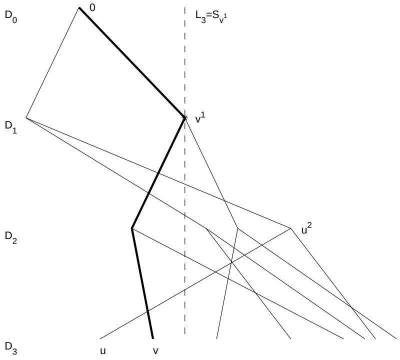
Outline

- Introduction of the model and result.
    - Branching random walk;
    - Minimal displacement;
    - Consistent Minimal displacement.
- Trials and errors leading to the proof.

Definition of Branching Random Walks

- Given a Galton-Watson tree $\mathbb{T} = (V, E)$ (with branching laws given by $\{p_k\}_{k=0}^{\infty}$) and i.i.d. random variables $\{X_{uv}\}_{uv \in E}$ associated to each edge uv in the tree. Then for each $v \in \mathbb{D}_n$ (all the vertices in the n$^{\text{th}}$ level), one defines $S_v = \sum_{k=0}^{n-1} X_{v^k v^{k+1}}$ where $v^0, v^1, \ldots, v^n$ is the ancestor of $v (= v^n)$ at the level $0, 1, \ldots, n$. Then $\{S_v | v \in V\}$ forms a branching random walk.



Assumptions

- b-ary tree.
- Large deviation assumption:

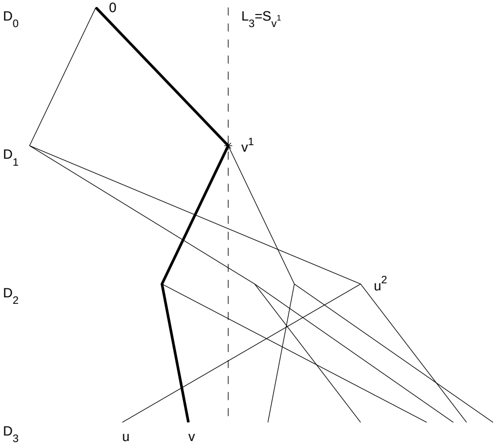
$$Ee^{\lambda X_e} < \infty \text{ for some } \lambda < 0 \text{ and some } \lambda > 0. \tag{1}$$

- Minimal displacement assumption: for some $\lambda_- < 0$ and $\lambda_+ > 0$ in the interior of $\{\lambda : \Lambda(\lambda) < \infty\}$, where $\Lambda(\lambda) = \log Ee^{\lambda X_e}$ is the log-moment generating function

$$\lambda_\pm \Lambda'(\lambda_\pm) - \Lambda(\lambda_\pm) = \log b, \tag{2}$$

Minimal Displacement

- Minimal displacement at level n: $m_n = \min_{v \in \mathbb{D}_n} S_v$.
- Under previous assumptions, $\lim_{n\to\infty} \frac{m_n}{n} = \Lambda'(\lambda_-) := m, \quad a.s.$.
- WLOG, by shift, we can and will assume $m = 0$ later on.



Consistent Minimal Displacement

- The offset is defined as $L_n = \min_{v \in \mathbb{D}_n} \max_{k=0}^{n} (S_{v^k} - mk)$ $(= \min_{v \in \mathbb{D}_n} \max_{k=0}^{n} S_{v^k}$ when $m = 0)$.

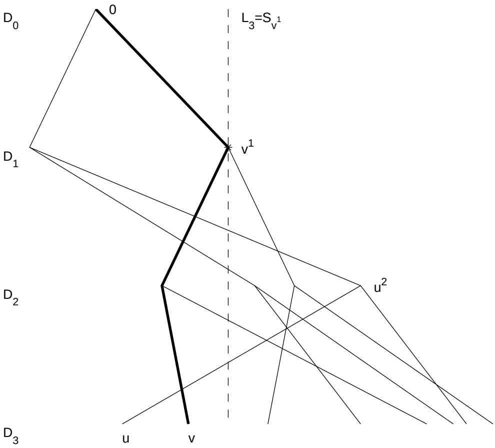


Figure: L_3 when $b = 2$ and $m = 0$

A Little More Comments about the Minimal Displacements

- More delicate results on m_n are available. That is, the second order term can be very different. For example, under different assumptions,
  - Bramson(1978): $\lim_{n\to\infty} m_n < \infty$ a.s.
  - Dekking and Host (1991): $\lim_{n\to\infty} \frac{m_n \log 2}{\log \log n} \to g$ a.s.
- Some properties of m_n do not necessarily require the independence of displacements of the children of the same parent. (The independence inherited from the tree is enough.) For example,
  - Dekking and Host (1991) proved the tightness of m_n when X_{uv}s are only assumed to be bounded.
  - Fang and Zeitouni (2010, in progress): tightness of m_n when X_{uv}s are only assumed to have left exponential tails.

Results about the Consistent Minimal Displacements

Theorem (O. Zeitouni, M. Fang, 2009)

Under assumption (1) and (2) and with $l_0 = \sqrt[3]{\frac{3\pi^2\sigma_Q^2}{-2\lambda_-}}$ *where* σ_Q^2 *is a certain variance, it holds that*

$$\lim_{n\to\infty} \frac{L_n}{n^{1/3}} = l_0 \quad a.s.\,. \tag{3}$$

Stories (History, or Development)

- In the process of studying random walks in random environments on trees, Hu and Shi (2007) discovered that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \leq \liminf_{n\to\infty} \frac{L_n}{n^{1/3}} \leq \limsup_{n\to\infty} \frac{L_n}{n^{1/3}} \leq c_2.$$

- As part of their study of RWRE on trees, G. Faraud, Y. Hu and Z. Shi (2009) independently obtained Theorem 1.

A Large Deviation Result of Mogul'skii

Define $S_n(t) = \frac{X_0+X_1+\cdots+X_k}{n^{1/3}}$ for $\frac{k}{n} \leq t < \frac{k+1}{n}$, $k = 0, 1, \ldots, n-1$,, where $X_0 = 0$ and $\{X_i\}_{i\geq 1}$ are iid with $E_Q(X_i) = 0$. Let $f_1(t)$ and $f_2(t)$ be two right-continuous and piecewise constant functions on $[0, 1]$.
$G = \cup_{0\leq t\leq 1}\{(f_1(t), f_2(t)) \times t\}$ is a region bounded by $f_1(t)$ and $f_2(t)$.

Theorem (Mogul'skii, 1974)

Under the above assumptions,

$$Q(S_n(t) \in G, t \in [0, 1]) = e^{-\frac{\pi^2\sigma_Q^2}{2}H_2(G)n^{1/3}+o(n^{1/3})},$$

where

$$H_2(G) = \int_0^1 \frac{1}{(f_1(t) - f_2(t))^2}dt.$$

A First Moment Argument — Hope for a Lower Bound

Consider

$$N_n = \sum_{v \in \mathbb{D}_n} 1_{\{-c_- n^{1/3} \le S_{v_k} \le c_+ n^{1/3} \text{ for } k=0,1,\dots,n\}}.$$

Calculate the first moment, and we have

$$EN_n = e^{(c_+ - \frac{\pi^2 \sigma_Q^2}{2(c_- + c_+)^2}) n^{1/3} + o(n^{1/3})}.$$

Notice that when $c_- \to \infty$, we can choose $c_+ \to 0$ and still have $EN_n \to 0$. This implies that

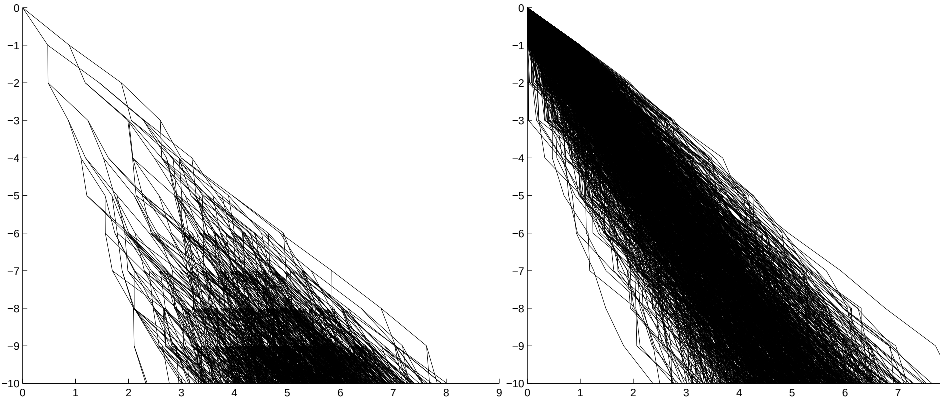
$$\liminf_{n \to \infty} \frac{L_n}{n^{1/3}} \ge 0.$$

It does NOT work!

Remark This first moment argument completely ignores the tree structure. In fact, if we define similar quantity L_n^{ind} for independent walks, we do have

$$\lim_{n \to \infty} \frac{L_n^{ind}}{n^{1/3}} = 0.$$

Difference between Branching Random Walks and Independent Random Walks



Recursion Inequality — some Lower Bound

Key Recursion Inequality

$$L_{m+n} \geq \min_{v \in \mathbb{D}_m} (S_v + L_n^v) \vee 0$$

where L_n^v is defined in the same way as L_n for each vertex $v \in V$.
Take exponentials first, use change of measure, and the best lower bound based on this recursion is 0.688 (for the standard Gaussian).

A Second Moment Argument — some Upper Bound

First and second moments are

$$EN_n = e^{(c_+ - \frac{\pi^2 \sigma_Q^2}{2(c_- + c_+)^2}) n^{1/3} + o(n^{1/3})}$$

and

$$EN_n^2 \leq e^{(cc_- + 2cc_+ - \frac{\pi^2 \sigma_Q^2}{2(c_- + c_+)^2}) n^{1/3} + o(n^{1/3})}.$$

Apply a second moment method, we obtain

$$P(N_n > 0) \geq \frac{(EN_n)^2}{EN_n^2} \geq e^{(-cc_- - \frac{\pi^2 \sigma_Q^2}{2(c_- + c_+)^2}) n^{1/3} + o(n^{1/3})}.$$

By a truncation (at level of order $n^{1/3}$) argument, we get some upper bound. In standard Gaussian case, the optimal truncation would give us an upper bound 3.047.

A Closer Look at the Second Moment Argument

- The fixed right bound is kind of determined by the problem. But to impose a fixed left bound $-c_- n^{1/3}$ for all levels is not natural.
- Instead of approximating the whole branching random walks by independent walks, we can try to divide branching random walks into several levels and to approximating branching random walks of depth ϵn by independent random walks.
- We can then consider walks who stay within $[\phi_k n^{1/3}, l n^{1/3}]$ for levels between $k\epsilon n$ and $(k+1)\epsilon n$.

The Optimization Problem

After a second moment method calculation, we need (consider the continuous ϕ_t)

$$\max_t\{-\phi(t) + \int_0^t \frac{c}{(l-\phi(u))^2} du\} \leq 0$$

to make the truncation argument work. With $w(t) = l - \phi(t)$, we need

$$l \geq \max_t\{w(t) + \int_0^t \frac{c}{w(u)^2} du\}.$$

Thus the best upper bound we can hope by this argument is

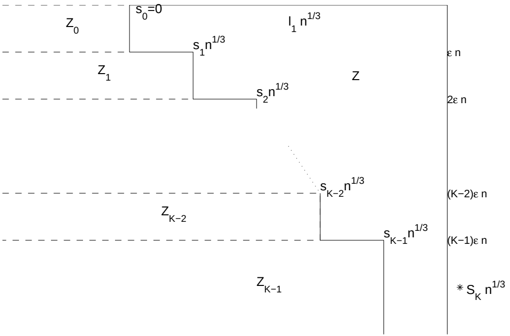
$$\min_{w:(0,1)\to\mathbb{R}_+} \max_t\{w(t) + \int_0^t \frac{c}{w(u)^2} du\},$$

and we solve this problem, we can find the 'best' curve $s(t)$ satisfies $s'(t) = -\frac{\pi^2\sigma_Q^2}{2\lambda_-(l-s(t))^2},\ s(0) = 0.$

The Lower Bound — the First Moment Revisited

In the picture, Z_i denote the number of vertices in the correspondence region. When $l_1 < l_0$, using result of Mogul'skii, we can prove that $E(\sum_{k=0}^{K-1} Z_k + Z) \to 0$ exponentially in $n^{1/3}$. Thus $\sum_{k=0}^{K-1} Z_k + Z = 0$ a.s. for all large n. That gives the lower bound

$$\liminf_{n\to\infty} \frac{L_n}{n^{1/3}} \geq l_0 \quad a.s.. \tag{4}$$



The Upper Bound — A Modified Second Moment Argument

When $l_2 > l_0$, define

$$\tilde{N}_n^{l_2} = \sum_{v \in \mathbb{D}_n} 1_{\{S_{v,j} \in [s_k n^{1/3}, l_2 n^{1/3}],\ \text{for } k\epsilon n \le j \le (k+1)\epsilon n,\ k=0,\ldots,\frac{1}{\epsilon}-1\}}.$$

Calculating $E\tilde{N}_n^{l_2}$ and $E(\tilde{N}_n^{l_2})^2$, we obtain by second moment method

$$P(\tilde{N}_n^{l_2} > 0) \ge P(\tilde{N}_n^{l_2} > 0) \ge e^{-\epsilon_2 n^{1/3} + o(n^{1/3})}$$

for some ϵ_2 small. This is good enough for us to obtain the upper bound by a standard truncation argument.

End of Presentation.

Thank You!

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Have a nice half-day break!