WHICH VECTOR FIELDS ON PATH SPACES ARE GRADIENTS?

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Report on work with Yuxin Yang

Sub-title

NON-ANTICIPATION AIDING GEOMETRY

Sub-subtitle

Vanishing of L^2 harmonic one forms on based path space

key-words: Malliavin calculus, Clark-Ocone formula, Markovian connections, exterior derivative

Starting point: Len Gross "Potential Theory on Hilbert Spaces" JFA 1967

Basic Question

For V in L^2:

When does there exist $f \in L^2$ with

$$grad\ f = V?$$

Poincare' Lemma for $\mathbf{R}^n$

$$\exists\, f \text{ with } grad\ f = V$$

$$\Longleftrightarrow$$

$$\langle DV(x)(a), b\rangle = \langle DV(x)(b), a\rangle$$

for all $a, b \in \mathbf{R}^n$, and $x \in \mathbf{R}^n$.

Set

$$d^1 V^\sharp_x(a,b) := d^1 V^\sharp(a \wedge b) := \frac{1}{2}\left(\langle DV(x)(a), b\rangle - \langle DV(x)(b), a\rangle\right)$$

Exterior derivative.

$\mathbf{R}^n$, $n = 1$. Ends

Take $g : \mathbf{R} \to \mathbf{R}$ with

$$g(x) = \begin{cases} -1 & \text{if } x \text{ on Left} \\ +1 & \text{if } x \text{ on Right} \end{cases}$$

Set $V = grad\ g$. Compact support , so $V \in L^2$.

$$d^1 V^\sharp = 0.$$

BUT $V \neq grad\ f$ for $f \in L^2$ since neither end has finite measure.

Related question

Do there exist harmonic functions of finite energy?i.e.

$$h : M \to \mathbf{R}$$

s.t.

1. $\Delta h = 0$
2. $V := grad\ h \in L^2$
3. h not constant.

Notation:

$$\Delta = \mathrm{div}\, grad = -d^* d$$

If so: $\mathrm{div}\ V = 0$ & $d^1 V^\sharp = 0$. Harmonic vector field.

L^2 Hodge Decomposition

M complete , μ smooth with positive density. Then:
The Hilbert space of L^2-vector fields

$$= \overline{\text{Image of } grad \text{ on } L^2\text{-functions}} \oplus \mathcal{H}^1_\mu(M)$$

$$\oplus \overline{\text{Image of } (div^1) \text{ on 2-vectors}}$$

where $\mathcal{H}^1_\mu(M) = \{V : d^1 V^\sharp = 0 \ \& \ \mathrm{div} \ V = 0\}$ is the space of L^2 Harmonic vector fields. Also div^1 is $(d^1)^*$.
Positive answer to our question iff *grad* has closed range and there are no L^2-harmonic vector fields:

$$\mathcal{H}^1_\mu(M) = 0.$$

$\mathbf{R}^n$

grad does not have closed range

Classical Wiener space

$$M = \Omega = C_0\mathbf{R}^m = \{\sigma : [0, T] \to \mathbf{R}^m : \text{ cts. \& s.t. } \sigma(0) = 0\}.$$

$$\mu = \mathbf{P} = \text{Wiener measure}$$

$$H = \text{Cameron-Martin Space} = L_0^{2,1}\mathbf{R}^m$$

$$\langle h^1, h^2\rangle_H = \int_0^T \langle \dot{h}^1(t), \dot{h}^2(t)\rangle_{\mathbf{R}^m} dt$$

As in Malliavin calculus we follow Gross: take H-valued vector fields, H-derivatives.

H-derivative and gradient

Closed operators:

$$d : \mathcal{D}^{2,1} \subset L^2(C_0\mathbf{R}^m; \mathbf{R}) \to L^2(C_0\mathbf{R}^m; H^*)$$

$$grad : \mathcal{D}^{2,1} \subset L^2(C_0\mathbf{R}^m; \mathbf{R}) \to L^2(C_0\mathbf{R}^m; H)$$

$$df_x(v) = \langle grad\ f(x), v \rangle_H$$

$$df_x = (grad\ f)^\sharp$$

$$grad\ f(x) = (df_x)^\sharp$$

exterior derivative:Shigekawa (1986)

Closed operator:

$$d^1 : Dom(d^1) \subset L^2(C_0\mathbf{R}^m; H^*) \to L^2(C_0\mathbf{R}^m \to (\wedge^2 H)^*)$$

defined for H-vector fields A,B with $A(x) = a$ and $B(x) = b$ by:

$$2d^1\phi_x(a\wedge b) = d\,(\phi(B(\cdot)))_x\,A(x) - d(\phi(A(\cdot)))_x B(x) - \phi_x([A,B](x))$$

For $V : C_0\mathbf{R}^m \to H$ we have

$$2d^1 V_x^\sharp(a \wedge b) = \langle dV_x(a), b\rangle_H - \langle dV_x(b), a\rangle_H.$$

Clark-Ocone Formula

If $f \in \mathcal{D}^{2,1}$ then

$$f = \int_{C_0 \mathbf{R}^m} f \, d\mathbf{P} - \mathrm{div}(\mathcal{P}\, grad f)$$

where $\mathcal{P} : L^2(C_0\mathbf{R}^m; H) \to L^2(C_0\mathbf{R}^m; H)$ is the projection onto adapted vector fields.
Martingale representation theorem

Consequences: *grad* and *d* have closed range
and $\mathcal{P} grad\ f = 0 \Leftrightarrow grad\ f = 0 \Leftrightarrow f = constant$

Beginning remark (Yuxin Yang)

Set

$$CO(V) = -\operatorname{div}(\mathcal{P}V) : C_0\mathbf{R}^m \to \mathbf{R}$$

If $df \in L^2$ then $f = c + CO(grad\ f) \in L^2$. In fact:

$$V^\sharp \in \text{Dom}(d^1) \Longrightarrow ||V - grad\ CO(V)||_{L^2} \le \sqrt{2}||d^1 V^\sharp||_{L^2}$$

Consequence: $d^1 V^\sharp = 0 \Leftrightarrow V = grad\ f$ some $f \in \mathcal{D}^{2,1}$
cf Shigekawa '86, Leandre '96

More precisely:

$$V^\sharp \in \mathrm{Dom}(d^1)$$

$$\Longrightarrow$$

$$\frac{d}{dt}(V_t - gradCO(V)_t) = \int_t^T \langle \mathbf{E}\{\frac{d}{ds}(grad\,\dot{V}_t)_s - \tau\frac{d}{dt}(grad\,\dot{V}_s)_t | \mathcal{F}_s\}, dB_s\rangle_{\mathbf{R}^m}$$

$$= -2\int_t^T {}_{(2)}\langle \mathbf{E}\{\frac{d}{dt}\otimes\frac{d}{ds}(d^1V^\sharp)^\sharp_{t,s} | \mathcal{F}_{s\vee t}\}, dB_s\rangle_{\mathbf{R}^m}$$

Consequence:
For some $f \in \mathcal{D}^{2,1}$
$d^1V^\sharp = 0 \Leftrightarrow V = grad\ f \Leftrightarrow \mathbf{E}\{(\frac{d}{dt}\otimes\frac{d}{ds})(d^1V^\sharp)^\sharp_{t,s} | \mathcal{F}_{s\vee t}\} = 0$

Based paths on M, compact connected Riemannian

$$C_{x_0}M = \{\sigma : [0, T] \to M \text{ cts. with } \sigma(0) = x_0\}$$

$\mu =$ Brownian motion measure.

Tangent space at the path σ:

$$T_\sigma C_{x_0}M = \{v : [0, T] \to TM : \text{cts.with } v_0 = 0 \;\&\; v_t \in T_{\sigma(t)}M\}.$$

Bismut tangent spaces

Need *H*-differentiation.

$$\mathcal{H}_\sigma = \{v \in T_\sigma C_{x_0}M : v_t = /\!/_t h_t \text{ with } h \in L^{2,1}T_{x_0}M\}$$

Then

$$f : C_{x_0}M \to \mathbf{R} \qquad df_\sigma = d_H f_\sigma : \mathcal{H}_\sigma \to \mathbf{R} \qquad \sigma \in C_{x_0}M$$

Take it closed, (Driver), with associated gradient and divergence operators, and self-adjoint $\Delta = \mathrm{div}\, grad$.

Clarke-Ocone formula

If $f \in \mathcal{D}^{2,1}$ then by S.Fang

$$f = \int_{C_{x_0}M} f \, d\mu - \mathrm{div}(\mathcal{P} grad \, f)$$

where $\mathcal{P} : L^2\Gamma\mathcal{H} \to L^2\Gamma\mathcal{H}$ is the projection onto adapted vector fields.

Therefore (Fang): d has closed range in the space of L^2-H-one-forms

$$\phi_\sigma : \mathcal{H}_\sigma \to \mathbf{R} \quad \text{with} \quad \int_{C_{x_0}M} |\phi_\sigma|^2_{\mathcal{H}_\sigma} \, d\mu(\sigma) < \infty$$

Our question for vector fields on $C_{x_0}M$.

$$d^1\phi(U(x),V(x)) = d(\phi(V(\cdot)))_x\,U(x) - d(\phi(U(\cdot)))_x\,V(x) - \phi_x([U,V](x))$$

TROUBLE!! The bracket of *H*-vector fields may not be an *H*-vector field.

In general cannot define a suitable closed exterior derivative operator d^1 from $L^2\Gamma\mathcal{H}^*$ to $L^2\Gamma\wedge^2\mathcal{H}^*$, (Léandre).

BUT: OK on classical Wiener space, (Shigekawa,1986); for *M* a compact Lie group, bi-invariant metric with *flat connection*, even for loops,(Fang& Franchi,1997). Also (Kusuoka,1991) for a different approach. See also (Jones & Leandre, 1991),& many papers of (Leandre) for non-L^2 theories.

Modified approach: K.D.E.&Xue-Mei Li

Perturb $\wedge^2\mathcal{H}$ to more suitable new Hilbert spaces $\mathcal{H}^2$ with continuous inclusion

$$\mathcal{H}^2_\sigma \hookrightarrow \wedge^2 T_\sigma C_{x_0}$$

For smooth cylndrical 1-forms ϕ set:

$$d^1_H\phi = d^1\phi|\mathcal{H}^2.$$

This is closable. Take its closure! Call it d^1

H-2-vectors and H-2-forms

For $u \in \wedge^2 TC_{x_0}$:

$$u \in \mathcal{H}^2 \iff u - \mathbb{R}(u) \in \wedge^2 \mathcal{H}$$
$$\iff u = (Id + Q)(v) \text{ some } v \in \wedge^2 \mathcal{H}$$

where $\mathbb{R} : \wedge^2 TC_{x_0} \to \wedge^2 TC_{x_0}$ is the curvature operator of the damped Markovian connection.
Get Hodge decomposition for L^2-H-one-forms.
K.D.E. & Xue-Mei Li, JFA 2007

Damped Markovian connection

For $V \in \mathcal{D}^{2,1}\mathcal{H}$ and $v \in \mathcal{H}_\sigma$ we have

$$\mathbb{V}_v V \in \mathcal{H}_\sigma$$

Define $\mathbb{V}^\sharp : \mathcal{D}^{2,1}\mathcal{H} \to L^2\Gamma(\mathcal{H} \otimes \mathcal{H})$ by

$$\langle \mathbb{V}^\sharp(V)_\sigma, v^1 \otimes v^2 \rangle_{\mathcal{H}_\sigma \otimes \mathcal{H}_\sigma} = \langle \mathbb{V}_{v^2} V, v^1 \rangle_{\mathcal{H}_\sigma}$$

Use of damped Markovian connection, $\breve{\nabla}$, on $\mathcal{H}$

Define $\mathfrak{D}^1 : \mathcal{D}^{2,1}(\mathcal{H}) \to L^2\Gamma(\wedge^2\mathcal{H})$ by

$$\mathfrak{D}^1(V) = \frac{1}{2}\left(\tau(\breve{\nabla}^\sharp V) - \breve{\nabla}^\sharp V\right) \in L^2 \wedge^2 \mathcal{H}.$$

Then for cylindrical ϕ:

$$d^1\phi(U) = \langle \mathfrak{D}^1\phi^\sharp, U\rangle_{\wedge^2\mathcal{H}} + \phi \boldsymbol{T}(U)$$

for the torsion $\boldsymbol{T} : \wedge^2\mathcal{H} \to TC_{x_0}M$ given by

$$\boldsymbol{T}(U^1 \wedge U^2) := \breve{\nabla}_{U^1} U^2 - \breve{\nabla}_{U^2} U^1 - [U^1, U^2].$$

Formula for Path Space

$\frac{\mathbb{D}}{dt}(grad\ CO(V) - V)_t = 2\int_t^T \langle \frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds}\mathcal{P}^{(2)}\mathfrak{D}^1 V, d\{\sigma_s\}\rangle_{\mathcal{H}}$
where
$\mathcal{P}^{(2)} : L^2\Gamma(\mathcal{H} \otimes \mathcal{H}) \to L^2\Gamma(\mathcal{H} \otimes \mathcal{H})$ is the projection on $\mathbf{A}^2$, the space of $U \in L^2\Gamma(\mathcal{H} \otimes \mathcal{H})$ such that $U_{s,t}$ is $\mathcal{F}_{s \vee t}$-measurable.

Consequence: $V = grad\ f$ some $f \in \mathcal{D}^{2,1}$ iff $\mathcal{P}^{(2)}\mathfrak{D}^1 V = 0$.
BUT $\mathfrak{D}^1 grad\ f = -(df \circ \boldsymbol{T})^*$

Q and $\boldsymbol{T}$

- $\operatorname{div} \boldsymbol{T}(U) = 0$ if $U = U^1 \wedge U^2$ with U^1 and U^2 adapted. {*Cruzeiro & Fang 1997*}.
- $\boldsymbol{T}(U) = \operatorname{div}^1 Q(U)$ if $U = U^1 \wedge U^2$ with U^1 and U^2 adapted. {*Elworthy &Xue-Mei Li 2008*}.
- Such $U = U^1 \wedge U^2$ are total in $\mathbf{A}^2$. Therefore the above hold for $U \in \mathbf{A}^2$.

$\mathfrak{D}^1$ and d^1

From this:

$$\mathcal{P}^2\mathfrak{D}^1(V) = \mathcal{P}^2(Id - \mathbb{R})(d^1 V^\sharp)^\sharp.$$

Revised formula; conclusions

If $V \in dom\, d^1$ then $CO(V) \in \mathcal{D}^{2,1}$ and

$$\frac{I\!\!D}{dt}(grad\ CO(V) - V) = 2\int_t^T \langle \frac{I\!\!D}{dt} \otimes \frac{I\!\!D}{ds}\mathcal{P}^{(2)}(d^1 V)^\sharp, d\{\sigma_s\}\rangle$$

. Thus:

- $V = grad\ f$ some $f \in \mathcal{D}^{2,1}$ iff $d^1 f = 0$.
- If ϕ is an L^2 harmonic H-one-form then $\phi = 0$.
- ?It seems that our definition of the exterior derivative was the correct one !

Based Loop Spaces

- Eberle: d does not have closed range if M has a closed geodesic with a neighbourhood of negative curvature

- Aida: d does have closed range for certain radially symmetric asymptotically flat manifolds

- X.Chen, Xue-Mei Li, Bo Wu: d does have closed range for n-dimensional hyperbolic space.

- No definition of d^1 for L^2 forms BUT for based loops on Lie groups G using left invariant connections all works (Fang &Franchi), and (Aida) gets L^2 cohomology vanishing for G simply connected.

end

THAT´S IT THANKS !