A continuum tree-valued Markov process

Jean-François Delmas

http://cermics.enpc.fr/~delmas

CERMICS, Univ. Paris-Est

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Outline

1. Discrete case
2. Continuous branching (CB) process

In collaboration with R. Abraham (Univ. Orléans, France).

Discrete case: Galton-Watson (GW) tree

- Poisson offspring distribution: $\mathcal{P}(\mathrm{e}^{-\theta_0})$. G-W tree: T_{θ_0}. It is
  - sub-critical if $\theta_0 > 0$: a.s. extinction,
  - critical if $\theta_0 = 0$: a.s. extinction,
  - super-critical if $\theta_0 < 0$: $\mathbb{P}(\text{extinction}) < 1$.
- Pruning of GW tree.
  - Add indep. exponential random variables τ (mean=1) on each branch.
  - Let $\theta > 0$. Cut all branches s.t. $\tau \leq \theta$.
  - The sub-tree $T_{\theta_0+\theta}$ (with the initial root) is a GW tree with offspring distribution $\mathcal{P}(\mathrm{e}^{-(\theta_0+\theta)})$.

Discrete tree-valued Markov process

- **Decreasing** tree-valued Markov process: $\theta \mapsto T_{\theta_0+\theta}$.
- Consistency allows to define $\theta \mapsto T_\theta$ for $\theta \in \mathbb{R}$.
- **Increasing** tree-valued Markov process (by time reversal): $\theta \mapsto T_{-\theta}$ for $\theta \in \mathbb{R}$.
- For $q < \theta$: transition kernel: $T_\theta \mapsto T_q$:
  - Let T_θ be given.
  - For each node $\mathbf{e}$ of T_θ let $N_\mathbf{e}$ be indep. Poisson r.v. with parameter $\mathrm{e}^{-q} - \mathrm{e}^{-\theta}$.
  - For each node $\mathbf{e}$ of T_θ attach $(T^{\mathbf{e},i}, i \in \{1, \ldots, N_\mathbf{e}\})$ indep. GW trees distributed as T_q.
  - Then: the big tree is distributed as T_q.

Explosion of (increasing) tree-valued Markov process

- Aldous and Pitman 1998: distribution of the **explosion time**

$$A = \inf\{\theta; \text{Card }(T_\theta) < +\infty\} \quad (< 0)$$

distribution of T_{A+} and of $(T_{A+q}, q > 0)$.

- For pruning at node (instead of branches) see Abraham, D. and He (2010) or next talk.

- **Remark**. Let $Z_\theta(n) = \text{Card }\{\text{population at time } n \text{ of } T_\theta\}$.
$Z_\theta = (Z_\theta(n), n \in \mathbb{N})$ is the so-called G-W process. Then for $q < \theta$,

$$Z_\theta + \text{ Immig. (at time } n\text{) of } \mathcal{P}\Big(Z_\theta(n)(\mathrm{e}^{-q} - \mathrm{e}^{-\theta})\Big) \text{ indep. G-W distrib. as } Z_q$$

is distributed as Z_q.

Continuous branching (CB) process

- Lamperti 1967: a CB $Y = (Y(t), t \in \mathbb{R}_+)$ is the limit of rescaled (mass and time) GW processes.
- $Y(t)$ is the "mass" of a population at time t whose individuals are of infinitesimal mass and have infinitesimal lifetime.
- ψ-CB Y is Markov with $Y(0) = x$ under $\mathbb{P}_x$ and

$$\mathbb{E}_x[\mathrm{e}^{-\lambda Y(t)}] = \mathrm{e}^{-xu(\lambda,t)} \quad \text{with} \quad \int_{u(\lambda,t)}^{\lambda} \frac{dv}{\psi(v)} = t.$$

- The excursion (canonical) measure is the "distribution" of the population coming from an infinitesimal individual; informally:

$$\mathbb{N}[Y \in A] = \lim_{x \to 0} \frac{1}{x} \mathbb{P}(Y \in A | Y(0) = x).$$

Quadratic CB

- For the talk: quadratic branching mechanism: $\beta \geq 0$ and

$$\psi(\lambda) = \beta\lambda^2.$$

- **Shifted branching mechanism**: for $\theta \in \mathbb{R}$,

$$\psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta) = \beta\lambda^2 + 2\beta\theta\lambda.$$

- The ψ_θ-CB, Y_θ, is a Feller diffusion:

$$dY_\theta(t) = \sqrt{2\beta Y_\theta(t)}\, dW_t - 2\beta\theta Y_\theta(t)\, dt.$$

Extinction for CB process

- Extinction: $\{Y_\theta(t) = 0 \text{ for } t \text{ large}\}$.
- We have: $\mathbb{E}_x[Y_\theta(t)] = x\, \mathrm{e}^{-\psi'_\theta(0)t}$.
  - CB sub-critical if $\psi'_\theta(0) > 0$ $(\theta > 0)$: a.s. extinction.
  - CB critical if $\psi'_\theta(0) = 0$ $(\theta = 0)$: a.s. extinction.
  - CB super-critical if $\psi'_\theta(0) < 0$ $(\theta < 0)$: $\mathbb{P}(\text{extinction}) < 1$.
- Total population mass: $\sigma_\theta = \int_0^\infty Y_\theta(s)\, ds$ and

$$\{\sigma_\theta < +\infty\} = \{\text{extinction}\}.$$

- Recall that $\mathbb{N}\left[1 - \mathrm{e}^{-\lambda\sigma_\theta}\right] = \psi_\theta^{-1}(\lambda)$. Thus:

$$\mathbb{N}[\sigma_\theta = +\infty] = \lim_{\lambda \downarrow 0} \mathbb{N}\left[1 - \mathrm{e}^{-\lambda\sigma_\theta}\right] = \psi_\theta^{-1}(0+) = \begin{cases} 0 & \text{if } \theta \geq 0, \\ 2\,|\theta| & \text{if } \theta < 0. \end{cases}$$

Growing family of CB

Let $q < \theta$.

- Consider a CB Y_θ under $\mathbb{N}$ (population with infinitesimal initial mass).
- Consider immigration of indep. ψ_q-CB:

$$(t_i, Y_q^i; i \in I)$$

  atoms of a Poisson point meas. with intensity (proportional to the size of the population) $2\beta Y_\theta(t)\ dt\ \mathbb{N}[dY_q]$.

- The process defined for $t \geq 0$ by

$$Y_\theta(t) + \sum_{i \in I; t_i > t} Y_q^i(t - t_i)$$

  is distributed as Y_q.

This allows to construct a family of CB-processes $(Y_\theta, \theta \in \mathbb{R})$ s.t. for all $t \geq 0$, $q < \theta$ we have,

$$Y_q(t) > Y_\theta(t) \quad \text{a.s.} \quad \text{and thus} \quad \sigma_q > \sigma_\theta.$$

Explosion time and results

- Explosion time defined by:

$$A = \inf\{q; \sigma_q < +\infty\} \quad (< 0).$$

- Let $\theta < 0$. We have:

$$\mathbb{N}[A > \theta] = \mathbb{N}[\sigma_\theta = +\infty] = \psi_\theta^{-1}(0+) = 2\,|\theta|\,.$$

  A is "distributed" under $\mathbb{N}$ as 2 times the Lebesgue meas. on $(-\infty, 0)$.
- We compute the distrib. of Y_{A+} and $(Y_{A+q}, q > 0)$ cond. on $\{A = \theta\}$.
- In particular, cond. on $\{A = \theta\}$, $(\sigma_{A+q}, q > 0)$ is distributed as

$$\left(\frac{1}{2\beta}\,\frac{1}{\tau_{|\theta|\,+q}}, q > 0\right),$$

  where τ is stable subordinator (index $1/2$): $\mathbb{E}[\exp -\lambda\tau_q] = \mathrm{e}^{-q\sqrt{2\lambda}}$.

Tools and more general results

- The genealogical tree (or Lévy tree or continuum random trees (CRT)) of CB process: Aldous (1991) for ψ_0, Duquesne and Le Gall (2002) for general **critical** or **sub-critical** ψ.
- Pruning CRT (marks on the skeleton and on the nodes): Abraham, D. and Voisin (preprint 2008). This allows to define a decreasing CRT-valued Markov process.
- Extension to **super-critical** cases using a Girsanov formula:

$$\frac{d\mathbb{P}^{\psi_\theta}_{|\mathcal{F}_t}}{d\mathbb{P}^{\psi}_{|\mathcal{F}_t}} = \exp\left(\theta Y(0) - \theta Y(t) - \psi(\theta)\int_0^t Y(s)\, ds\right),$$

  where Y is a ψ_θ-CB under $d\mathbb{P}^{\psi_\theta}$ and $\psi_0 = \psi$ is critical.
  (**Also true for CRT and general** ψ.)
- Use consistency and time reversal to define the (increasing) tree-valued Markov process.
- Results: distribution of the CRT at the explosion time for general ψ.