Some Results Evolutionary Prisoner's Dilemma Games

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Contents

- Prisoner's dilemma games for 2 players.
- Any way out of the dilemma?
- Our model: local interaction with mutation for $n \geq 5$ players.
  like 1-dim interaction particle system
- Dynamics I : Rational strategy for next time period by
  imitating-most-successful-player, or
  imitating-most-successful-action
- Dynamics II : Mutation
- Jointed works with H.C. Chen and L.D. Wu.

Prisoner's Dilemma Game

- 2 isolated prisoners to be sentenced.
- Strategy set { Defect, Cooperation }. Like spin $\{\pm\}$.
- Payoffs:

	D	*C*
D	*6 years, 6 years*	*3 months, 10 years*
C	*10 years, 3 months*	*1 year, 1 year*

- Nash Equilibrium is (D, D).
  But (C, C) is better.
- Payoff for strategy D $>$ payoff for strategy C.
- Any way out of the dilemma?
- Karandikar et al. (1998), Palomino and Vega-Redonda (1999), Ellison (1993), Eshel et al. (1998) and so on.

Prisoner's Dilemma Game continued...

With $b > d > a > c$, the payoff in general is

	D	C
D	a, a	b, c
C	c, b	d, d

- Nash Equilibrium is (D, D). But (C, C) is better.
- Payoff for strategy D $>$ payoff for strategy C.
- Definition. (s, t) is a Nash equilibrium if

$$\text{payoff at } (s, t) \geq \text{ payoff at } (s, t') \quad \forall t' \in S;$$

$$\text{payoff at } (s, t) \geq \text{ payoff at } (s', t) \quad \forall s' \in S.$$

  I.e., no player gains by changing his present strategy individually.
- New models: many players, many times, local structure.

Evolutionary games with local interaction

Similar to interacting particle systems.

- $N = \{1,\ 2, \ldots, n\},\ \ n \geq 5$, be the set of players.
- 1-dim setup: Players sit sequentially around a circle.
- NN interaction: $N_i = \{i-1,\ i+1\}$ is the set of player i's neighbors.
- Let $\vec{s} = (s_1, s_2, ..., s_n)$ be the strategy profile at time t. Here, $s_i \in \{C, D\}$ for each player i.
- The dynamics for forming the strategy for time $t+1$ consists of 2 parts.

Dynamics I. Strategy revision by imitation

Each player imagines to play the above PD game once with each of their two neighbors.
Let $z_i(\vec{s})$ = player i's total payoff thus incurred. Then

$$z_i(\vec{s}) = \begin{cases} b \cdot n_i^C(\vec{s}) + a \cdot (2 - n_i^C(\vec{s})) & \text{if } s_i = D, \\ d \cdot n_i^C(\vec{s}) + c \cdot (2 - n_i^C(\vec{s})) & \text{if } s_i = C. \end{cases}$$

Here $n_i^C(\vec{s}) = |\{j \in N_i : s_j = C\}|$ is the number of player i's neighbors taking strategy C at time t.

- Imitating-most-successful-player in his neighborhod: the rational choice for player i is

$$r_i(\vec{s}) \in M_i(\vec{s}) \stackrel{\text{def}}{=} \{s_j : z_j(\vec{s}) = \max z_k(\vec{s}) \text{ for } k \in N_i \cup \{i\} \}.$$

Dynamics I. continued...

- Imitating-most-successful-action: each player i will imitate the most successful action yielding the highest average payoff which was adopted among his neighbors and himself at time t. Let δ be the Kronecker notation. Then

$$a_i(\vec{s}) = \begin{cases} \frac{\sum_{k \in N_i \cup \{i\}} z_k(\vec{s}) \cdot \delta_{E,s_k}}{\sum_{k \in N_i \cup \{i\}} \delta_{E,s_k}}, & \text{if } E \in \{s_{i-1}, s_i, s_{i+1}\}, \\ -\infty, & \text{if } E \neq s_{i-1} = s_i = s_{i+1}, \end{cases}$$

means the average payoff for strategy $E \in \{C, D\}$ among player i and his neighbors. Therefore, player i's next-period rational choice $r_i(\vec{s})$ satisfies

$$r_i(\vec{s}) \in \bar{M}_i(\vec{s}) \stackrel{\text{def}}{=} \{E \in \{C, D\} : a_i^E(\vec{s}) = \max(a_i^C(\vec{s}), a_i^D(\vec{s}))\ \}.$$

Dynamics I. continued...

- The computation of $M_i(\vec{s})$ and $\bar{M}_i(\vec{s})$ for player i involves

  $$(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2})$$

  14 out of 32 cases need to be considered,

  $$\text{like } r_i(\vec{s}) = s_i \text{ if } s_{i-1} = s_i = s_{i+1}.$$

- For brevity, $r(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}) \stackrel{\text{def}}{=} r_i(\vec{s})$.
- Strict rule by inertia:

  $$r_i(\vec{s}) = s_i \text{ iff } s_i \in M_i(\vec{s}) \text{ (or } s_i \in \bar{M}_i(\vec{s})).$$

- Essentially the same results for the loose rule.
- A time-homogeneous Markov chain on $S = \{C, D\}^n$ with transition probability matrix $Q_0(\vec{s}, \vec{u}) = 1$ iff $\vec{u} = \vec{r}(\vec{s})$, where the rational choice $\vec{r}(\vec{s}) = (r_1(\vec{s}), r_2(\vec{s}), \ldots, r_n(\vec{s}))$ is uniquely determined for state $\vec{s} \in S$ by the strict rule.

Dynamics II. Mutation

Players will simultaneously, but independently alter their rational choices $\{r_i(\vec{s})\}$ with identical probability $\epsilon > 0$.
The mutation rate can be regarded as the probability of players' experimenting with new strategies.

All together, our local-interaction imitation dynamics define a Markov chain $\{X_t : t = 0, 1, ...\}$ on S.
Its transition matrix Q_ϵ, a perturbation of Q_0, given by

$$Q_\epsilon(\vec{s}, \vec{u}) = \epsilon^{d(\vec{r}(\vec{s}),\, \vec{u})} \cdot (1 - \epsilon)^{n - d(\vec{r}(\vec{s}),\, \vec{u})} \text{ for all } \vec{s}, \vec{u} \in S.$$

Here, $d(\vec{r}(\vec{s}), \vec{u}) = |\{i \in N : r_i(\vec{s}) \neq u_i\}|$ is the number of mismatches between the next truly-adopted strategy $\vec{u}$ and the revised rational choice $\vec{r}(\vec{s})$ at state $\vec{s}$.

- $U(\vec{s}, \vec{u}) = d(\vec{r}(\vec{s}), \vec{u})$ means the cost from $\vec{s}$ to $\vec{u}$.

Dynamics II. continued...

- $Q_\epsilon(\vec{s}, \ \vec{u}) > 0$ for all $\vec{s}, \ \vec{u} \in S$.
- Mutation makes our dynamic process $\{X_t\}$ ergodic.
- The unique invariant distribution μ_ϵ is characterized by

$$\mu_\epsilon = \mu_\epsilon \cdot Q_\epsilon .$$

- Goal: to find $\mu_* \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \mu_\epsilon$.
- In particular, whether

$$\vec{C} \in S_* \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \mu_*(\vec{s}) > 0\}?$$

I.e. whether all-cooperation is possible in the long run?

- Elements in S_* are called the Long Run Equilibria.

Method of Freidlin and Wentzell

- Letting $\epsilon \downarrow 0$ in $\mu_\epsilon = \mu_\epsilon \cdot Q_\epsilon$,
  Vega-Redondo (2003) showed $\mu_* = \mu_* \cdot Q_0$. Hence,

$$S_* \subseteq S_0 = \{\text{ all invariant states under } Q_0\}.$$

- We will first characterize S_0.
- Use the method of Freidlin and Wentzell to find S_* and the order estimate for $E_\epsilon(T)$, where

$$T = \inf\{t \geq 0 : X_t \in S_*\}$$

  is the waiting time to hit the global minimum set S_*.
- In case $U(\vec{s}, \vec{u}) = (U(\vec{u}) - U(\vec{s}))^+$, then
  $S_* = \{\vec{s} : U(\vec{s}) = \min U\}$.

Method of Freidlin and Wentzell continued...

- For any $\vec{s} \in S$, let

$$G(\{\vec{s}\}) = \{ \text{ all spanning trees rooted at } \vec{s}\}.$$

- $U(\vec{s}, \vec{u}) = d(\vec{r}(\vec{s}), \vec{u})$ means the cost from $\vec{s}$ to $\vec{u}$.
- $v(g) = \sum_{(\vec{u} \to \vec{v}) \in g} U(\vec{u}, \vec{v})$ means the cost of $g \in G(\{\vec{s}\})$.
- $v(\{\vec{s}\}) = \min_{g \in G(\{\vec{s}\})} v(g)$
  the minimum cost of all spanning trees rooted at $\vec{s}$.
- Define $v_1 = \min_{\vec{s} \in S} v(\{\vec{s}\})$:
  the minimum cost to build a network with 1 center.
- Then $\mu_* = \lim_{\epsilon \to 0} \mu_\epsilon$ exists and the following holds.

Method of Freidlin and Wentzell continued...

- **Theorem 1.** The support S_* *of* μ_* is given by

$$S_* = \{\vec{s} \in S \mid v(\{\vec{s}\}) = v_1\}$$

and $\mu_\epsilon(\vec{u}) \approx \epsilon^{v(\{\vec{u}\})-v_1}$ for any $\vec{u} \in S$.

- S_* consists of those states in S which attain the minimum cost v_1 when treated as a root.
- Let $G(W) = \{$ all spanning trees rooted at $W \subset S\}$ and $v(W) = \min_{g \in G(W)} v(g)$. Define

$$v_k = \min_{|W|=k} v(W) \text{ for } k \geq 1.$$

- **Theorem 2. (Chiang and Chow (2007))**

$$E_\epsilon(T) \approx \epsilon^{-\delta} \text{ as } \epsilon \downarrow 0.$$

Here $\delta = v_{k_0-1} - v_{k_0}$ and $k_0 = \min\{k \geq 2 : \exists W \subseteq S$ with $|W| = k, v(W) = v_k$ and $W \not\subseteq S_*\}$.

Results

- $M \stackrel{\text{def}}{=} S_0 \setminus \{\vec{C}, \vec{D}\}$
  is called the set of mixed stationary states, which means cooperators and defectors coexist peacefully.
- For $\vec{s} \in M \neq \emptyset$ can be expressed as follows:

$$\cdots \underbrace{D \cdots D}_{d_k} \underbrace{C \cdots C}_{c_k} \underbrace{D \cdots D}_{d_1} \underbrace{C \cdots C}_{c_1} \underbrace{D \cdots D}_{d_2} \underbrace{C \cdots C}_{c_2} \cdots$$

  d_i = length of the ith D-string,
  c_j = length of the jth C-string starting from a certain player.
- For positive integers m and ℓ, define

$$M_{\geq m,\, \geq \ell} \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } d_i \geq m,\ c_j \geq \ell\}$$

$$M_{m,\, \ell} \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } d_i = m,\ c_j = \ell\}.$$

Results continued...

Theorem 3. For Imitating-Successful-Player dynamics,
$S_* = \{\vec{D}\}$ and $E_\epsilon(T) \approx \epsilon^{-1}$ *as* $\epsilon \downarrow 0$.
If $a + b > 2d$, then $S_0 = \{\vec{C}, \vec{D}\}$;
If $a + b \leq 2d$, then $S_0 = \{\vec{C}, \vec{D}\} \cup M_{\geq 2, \geq 3}$.

- All-defection $\vec{D}$ is the unique LRE of the ISP dynamics. Yet S_0 depends on whether $a + b \leq 2d$ or not.
- Because

$$P(r(*, C, D, C, *) = D) = 1$$

and

$$P(r(*, D, C, D, *) = D) = 1,$$

which shows the strength of D against C.

Results continued...

Theorem 4. Assume the Imitating-Successful-Action dynamics.
(i) If $a+b > \frac{c+3d}{2}$, $S_0 = \{\vec{C}, \vec{D}\}$, $S_* = \{\vec{D}\}$ and $E_\epsilon(T) \approx \epsilon^{-1}$.
(ii) If $a+b \leq \frac{c+3d}{2}$ and $\frac{3a+b}{2} < c+d$, then $S_0 = \{\vec{C}, \vec{D}\} \cup M$, where the mixed stationary states in M has all $d_i \in \{1,\ 2,\ 3\}$ and, besides $c_i \geq 3$,

$$c_i \geq 5 \text{ if } (d_i, d_{i+1}) = (1,1);\ c_i \geq 4 \text{ if } (d_i, d_{i+1}) = (1,2) \text{ or } (2,1).$$

$$\begin{cases} S_* & = \{\vec{D}\} \text{ and } E_\epsilon(T) \approx \epsilon^{-1} \text{ for } n = 5, \\ S_* & = \{\vec{D}\} \text{ and } E_\epsilon(T) \approx \epsilon^{-\lceil \frac{n}{10} \rceil} \text{ for } 6 \leq n \leq 20, \\ S_* & = S_0 \text{ and } E_\epsilon(T) \approx \epsilon^{0} \text{ for } 21 \leq n < 30 \text{ but } n \neq 25, \\ S_* & = S_0 \setminus M_{2,\,3} \text{ and } E_\epsilon(T) \approx \epsilon^{-1} \text{ for } n = 25 \text{ or } 30, \\ S_* & = (S_0 \setminus M_{2,\,3}) \setminus \{\vec{D}\} \text{ and } E_\epsilon(T) \approx \epsilon^{-3} \text{ for } n \geq 31. \end{cases}$$

(iii) If $a+b \leq \frac{c+3d}{2}$ and $\frac{3a+b}{2} \geq c+d$, then
$S_0 = \{\vec{C}, \vec{D}\} \cup M_{\geq 2,\, \geq 3}$, $S_* = \{\vec{D}\}$ and $E_\epsilon(T) \approx \epsilon^{-1}$.

Results continued...

- Theorem 4 (ii) shows that when $a + b \leq \frac{c+3d}{2}$ and $\frac{3a+b}{2} < c + d$, S_* varies as the population size n grows from $\{\vec{D}\} = S_*$ for $5 \leq n \leq 20$ to $\{\vec{D}, \vec{C}\} \subset S_*$ for $21 \leq n \leq 30$, and finally to $S_* = (S_0 \setminus M_{2,\,3}) \setminus \{\vec{D}\}$ for $n \geq 31$. In particular, all-cooperation $\vec{C}$ instead of all-defection $\vec{D}$ becomes a LRE under the ISA dynamics when $\#$ of players ≥ 31.
- For positive integers m and ℓ, define

$$M_{\geq m,\,\geq \ell} \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } d_i \geq m,\ c_j \geq \ell\}$$

$$M_{m,\,\ell} \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } d_i = m,\ c_j = \ell\}.$$

- Chen and Chow, Evolutionary prisoner's dilemma games with local interaction and imitation, Adv. Applied Probab. 41(2009).

Match v rounds randomly

- In the above, each player plays the PD game once with each of his neighbors for strategy updating.
- What if players are randomly matched to play with his neighbors for v times?
- Only 2 ways to do the matching:

$$m_1 : 1 \leftrightarrow 2, 3 \leftrightarrow 4, ..., n-1 \leftrightarrow n,$$

$$m_2 : n \leftrightarrow 1, 2 \leftrightarrow 3, ..., n-2 \leftrightarrow n-1.$$

- Number of players n has to be even.
- By LLN, $v = \infty \Leftrightarrow$ plays once with each of his neighbors.
- **Theorem 5.** For both the ISP and ISA dynamics, $S_* = \{\vec{D}\}$ for any $1 \leq v < \infty$.
- Chow and Wu (2010), in preparation.

Match v rounds randomly continued...

- Theorem 4 (ii) shows that when $a + b \leq \frac{c+3d}{2}$ and $\frac{3a+b}{2} < c + d$, S_* varies as the population size n grows from $\{\vec{D}\} = S_*$ for $5 \leq n \leq 20$ to $\{\vec{D}, \vec{C}\} \subset S_*$ for $21 \leq n \leq 30$, and finally to $S_* = (S_0 \setminus M_{2,\,3}) \setminus \{\vec{D}\}$ for $n \geq 31$.
- Any mixed stationary state $\vec{s}$ in $M = S_0 \setminus \{\vec{C}, \vec{D}\}$ has all $d_i \in \{1,\ 2,\ 3\}$ and, besides $c_i \geq 3$,

$$c_i \geq 5 \text{ if } (d_i, d_{i+1}) = (1, 1);\ c_i \geq 4 \text{ if } (d_i, d_{i+1}) = (1, 2) \text{ or } (2, 1).$$

- Decompose M as $\cup_1^L M_k$, where $M_k \stackrel{\text{def}}{=} \{\vec{s} \in M : \vec{s} \text{ has } k \text{ disjoint } D\text{-strings }\}$.
- $\vec{s} \xrightarrow{k} \vec{u}$ means $U(\vec{s}, \vec{u}) = k$ and $\vec{s} \xleftrightarrow{k} \vec{u}$ if $U(\vec{u}, \vec{s}) = k$ as well.

Match v rounds randomly continued...

$$\vec{s} = \cdot\cdot \underbrace{DD}_{2}\,\underbrace{C\cdot\cdot C}_{c_{i-1}}\,\underbrace{\dot{D}}_{1}\,\underbrace{C\cdot\cdot C}_{c_i}\,\underbrace{DD}_{2}\cdot\cdot \overset{0}{\leftrightarrow} \cdot\cdot \underbrace{DD}_{2}\,\underbrace{C\cdot\cdot C}_{c_{i-1}-1}\,\underbrace{D\,\dot{D}\,D}_{3}\,\underbrace{C\cdot\cdot C}_{c_i-1}\,\underbrace{DD}_{2}\cdot\cdot$$

$$= \vec{s}_d \overset{1}{\to} \cdot\cdot \underbrace{DD}_{2}\,\underbrace{C\cdot\,\dot{C}\,\cdot C}_{c_{i-1}+1+c_i}\,\underbrace{DD}_{2}\cdot\cdot = \vec{u} \overset{1}{\to} \vec{s}.$$

- $\vec{s} \overset{1}{\leftrightarrow} \vec{u}$ for any $\vec{s}, \vec{u} \in M_k \setminus M_{2,\,3}$ for $k \geq 1$.
- $\vec{s} \overset{1}{\leftrightarrow} \vec{u}$ for any $\vec{s} \in M_k \setminus M_{2,\,3}$ and $\vec{u} \in M_{k-1}$ for $k \geq 1$. Here $M_0 = \{\vec{C}\}$.
- Any two states in $\{\vec{C}\} \cup M \setminus M_{2,\,3}$ are equivalent.

Match v rounds randomly continued...

- $\vec{D}$ can reach out at the minimum cost 3 as follows :

$$\vec{D} \xrightarrow{3} \underbrace{\overset{\circ}{C}\overset{\circ}{C}\overset{\circ}{C}}_{3}\underbrace{D\cdot\cdot D}_{n-3} \xrightarrow{0} \underbrace{C\cdot\cdot C}_{5}\underbrace{D\cdot\cdot D}_{n-5}$$

$$\xrightarrow{0} \underbrace{C\cdot\cdot C}_{7}\underbrace{D\cdot\cdot D}_{n-7} \xrightarrow{0} \cdots \xrightarrow{0} \vec{u} \in M_1,$$

  where the unique D-string in $\vec{u}$ has length 1 or 2 depending on n is even or odd.
- Let $\eta = \#$ of closed connected components in M.
- $v(\{\vec{s}\}) = 3 + \eta$ for any $\vec{s} \in \{\vec{C}\} \cup M \setminus M_{2,\,3}$.
  $v(\{\vec{u}\}) = 4 + \eta$ for any $\vec{u} \in M_{2,\,3}$
- In order to find the minimum cost path from $\vec{C}$ to $\vec{D}$, it suffices to do so from any $\vec{s} \in \{\vec{C}\} \cup M \setminus M_{2,\,3}$. And it saves to use some state with as many D's as possible.
  Note that any D-string in $\vec{s} \in M$ has length ≤ 3.

Match v rounds randomly continued...

If $d_i = d_{i+1} = 1$, it saves to have $c_i = 9$ as the ith C-string of $\vec{s}$ can be eliminated at cost 1 :

$$\vec{s} \overset{0}{\leftrightarrow} \vec{s}_d \overset{1}{\rightarrow} \cdots * \underbrace{\overset{\bullet}{D}}_{1} \underbrace{CCCC}_{4} \underbrace{\overset{\circ}{D}}_{1} \underbrace{CCCC}_{4} \underbrace{\overset{\bullet}{D}}_{1} * \cdots$$

$$\overset{0}{\rightarrow} \cdots \underbrace{D\overset{\bullet}{D}D}_{3} \underbrace{CC}_{2} \underbrace{DDD}_{3} \underbrace{CC}_{2} \underbrace{D\overset{\bullet}{D}D}_{3} \cdots$$

$$\overset{0}{\rightarrow} \cdots * \underbrace{\overset{\bullet}{D}}_{1} CDDCDCDDC \underbrace{\overset{\bullet}{D}}_{1} * \cdots \overset{0}{\rightarrow} \cdots \underbrace{D\overset{\bullet}{D}D \cdots D\overset{\bullet}{D}D}_{13} \cdots.$$

Hence, $v(\{\vec{D}\}) = \lceil \frac{n}{10} \rceil + \eta$.

Remember $v(\{\vec{s}\}) = 3 + \eta$ for $\vec{s} \in \{\vec{C}\} \cup M \setminus M_{2,\,3}$.

Theorem 4 (ii) then follows by comparing $\lceil \frac{n}{10} \rceil$ with 3.

Match v rounds randomly continued...

Proposition 6. Assume $v = 1$. For both ISP and ISA dynamics, we have $S_{0,v} = \{\vec{C}, \vec{D}\} \cup M^{odd}_{\geq 3,\geq 3}$ and $S_{*,v} = \{\vec{D}\}$. Here,

$$M^{odd}_{\geq 3,\geq 3} \stackrel{\text{def}}{=} M_{\geq 3,\geq 3} \bigcap \{\vec{s} \in S \mid \text{ all } c_i \text{ and } d_j \text{ are odd } \}.$$

- $\vec{D}$ can be reached at the minimum cost 1 as follows :

$$M_1 \ni \underbrace{C\,\overset{\circ}{C}\,C}_{3}\underbrace{D\cdot\cdot D}_{n-3} \xrightarrow{1} \underbrace{C\,\overset{\circ}{D}\,C}_{3}\underbrace{D\cdot\cdot D}_{n-3} \xrightarrow{0} \vec{D}$$

$$\vec{C} \xrightarrow{1} \cdots C\underbrace{\;D\;}_{1}C\cdots \xrightarrow{0} \cdots C\underbrace{DDD}_{3}CC\cdots \in M_1.$$

$$\vec{D} \xrightarrow{2} \cdots DD\underbrace{\overbrace{CC}^{\text{match}}}_{2}\overbrace{DD}\cdots \xrightarrow{0} \cdots D\underbrace{\overbrace{CC}^{\text{match}}\overbrace{CC}}_{4}DD\cdots \xrightarrow{0} \cdots \xrightarrow{0} \vec{C}.$$

- Easy to show $v(\{\vec{D}\}) = \eta + 1,\ v(\{\vec{C}\}) = \eta + 2$ and $v(\{\vec{s}\}) \geq \eta + 2$ for $\vec{s} \in M$.