Limit Theorems for Functions of Marginal Quantiles

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Joint work with

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Outline

1. Introduction
2. Main Results
3. Proofs (Sketch)
4. Examples/Counterexamples

1. INTRODUCTION

DeGroot & Goel (1980) considered

- random sample of size n drawn from a bivariate normal,
- before observations (x_i, u_i)'s, the association between x_i and u_i are lost.
- Observations are then only available in the following sense:
  - $x_1, \dots, x_n$ and $y_1, \dots, y_n$, where
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- Interested in estimating the correlation coefficient, ρ

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- Our motivation: Estimating the parameters in a linear regression problem $y = \alpha + \beta x$ for (X_i, Y_i)'s when the association between the X_i's and the Y_i's is lost.
- If the linkage were not lost, elementary fact:

$$\hat{\beta} = (\sum_{i=1}^{n} X_i Y_i - n\bar{X}\bar{Y}) / \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

- Natural candidate for estimating β when association is lost: Average $\sum_{i=1}^{n} X_i Y_{\pi(i)}$ over all permutations π.
- That is,

$$\frac{1}{n!} \sum_{\pi} \left(\frac{1}{n} \sum_{i=1}^{n} X_i Y_{\pi(i)} \right)$$

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$$X_{n:1} \leq X_{n:2} \leq \cdots \leq X_{n:i}$$

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$$\frac{1}{n!}\sum_{\pi}\left(\frac{1}{n}\sum_{i=1}^{n} X_i Y_{\pi(i)}\right) \leq \frac{1}{n}\sum_{i=1}^{n} X_{n:i} Y_{n:i}$$

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- Consider a general problem: asymptotic behavior of

$$\frac{1}{n}\sum_{i=1}^{n}\phi(X_{n:i}, Y_{n:i})$$

- Results can be generalized to d-dimension:

  Let $(X_i^{(1)}, \ldots, X_i^{(d)})$, $1 \le i \le n$, be IID random vectors.

  For each $1 \le j \le d$, order $X_i^{(j)}$, $1 \le i \le n$:

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2. Main Results

Some notation:

- $(X_1, Y_1), \ldots, (X_n, Y_n)$ be IID, with joint df H:
  - $X_1, X_2, \ldots, X_n \sim F$.
  - $Y_1, Y_2, \ldots, Y_n \sim G$
  - Define $F^{-1}(y) = \inf\{x : F(x) \geq y\}$.
- **Theorem 1** *Let ϕ be a real-valued measurable function on $\mathbb{R}^2$ satisfying some conditions, then, as $n \to \infty$,*

$$\frac{1}{n}\sum_{i=1}^{n} \phi(X_{n:i}, Y_{n:i}) \xrightarrow{a.s.} \bar{\gamma}.$$

Here

$$\bar{\gamma} := \int_0^1 \phi(F^{-1}(y), G^{-1}(y))dy = E\phi\left(F^{-1}(U), G^{-1}(U)\right)$$

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Theorem 2 *Let ϕ be a real-valued measurable function on $\mathbb{R}^2$ satisfying some further conditions, then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi(X_{n:i},Y_{n:i})-\sqrt{n}\,\bar{\gamma}=\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n}Z_{n,\ell}+o_P(1)$$

where

$$Z_{n,\ell}=\frac{1}{n}\sum_{i=1}^{n}\left\{\psi_1(\frac{i}{n+1})[I(U_\ell\le\frac{i}{n})-\frac{i}{n}]+\psi_2(\frac{i}{n+1})[I(V_\ell\le\frac{i}{n})-\frac{i}{n}]\right\}.$$

Here $U_\ell=F(X_\ell)$, $V_\ell=G(Y_\ell)$, $\psi_1(u)=\frac{\partial\phi(F^{-1}(x),G^{-1}(y))}{\partial x}|_{(x,y)=(u,u)}$ and ψ_2 is similarly defined.

Theorem 2 (contd.) *Further, as $n \to \infty$,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi(X_{n:i}, Y_{n:i}) - \sqrt{n}\, \bar{\gamma} \xrightarrow{dist} N(0, \sigma^2)$$

where

$$\begin{aligned} \sigma^2 &= 2 \int_0^1 \int_0^v u(1-v)[\psi_1(u)\psi_1(v) + \psi_2(u)\psi_2(v)]\, dudv \\ &\quad +2 \int_0^1 \int_0^1 [K(u,v) - uv]\psi_1(u)\psi_2(v)\, dudv. \end{aligned}$$

Here $K(u,v) = H(F^{-1}(u), G^{-1}(v))$.

3. Sketches of Proofs

Proof (sketch) of Theorem 1

- Key observation

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\phi(X_{n:i},Y_{n:i}) &\approx \frac{1}{n}\sum_{i=1}^{n}\phi\left(F^{-1}(\frac{i}{n+1}),G^{-1}(\frac{i}{n+1})\right)\\ &\approx \int_0^1 \phi\left(F^{-1}(y),G^{-1}(y)\right)\,dy.\end{aligned}$$

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Proof (sketch) of Theorem 2

- Introduce $U_i = F(X_i)$ and $V_i = G(Y_i)$ for $1 \le i \le n$.
- Joint distribution of (U_1, V_1) is $K(u,v) = H(F^{-1}(u), G^{-1}(v))$.
- Denote $\psi(u,v) = \phi(F^{-1}(u), G^{-1}(v))$.
- Key approximations, with $\mu_{n:i} = i/(n+1)$,

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\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi(X_{n:i}, Y_{n:i}) - \sqrt{n}\,\bar{\gamma} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(U_{n:i}, V_{n:i}) - \sqrt{n}\,\bar{\gamma} \\
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&\approx \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{\ell=1}^{n} \left\{ [I(U_\ell \le \frac{i}{n}) - \frac{i}{n}]\psi_1(\mu_{n:i}) + [I(V_\ell \le \frac{i}{n}) - \frac{i}{n}]\psi_2(\mu_{n:i}) \right\}
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$$\approx \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\psi(U_{n:i}, V_{n:i}) - \psi(\mu_{n:i}, \mu_{n:i})]$$

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\approx \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{\ell=1}^{n} \left\{ [I(U_\ell \leq \frac{i}{n}) - \frac{i}{n}]\psi_1(\mu_{n:i}) + [I(V_\ell \leq \frac{i}{n}) - \frac{i}{n}]\psi_2(\mu_{n:i}) \right\}
\end{aligned}
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4. Examples/Counterexamples

Example 1 Some condition on ϕ is needed.

- Let
$$\phi(x,y) = I(x = y).$$
- Let $X_1, \ldots, X_n,\ Y_1, \leq Y_n$ be independent with common uniform distribution on (0, 1).
- Note that
  - $P(X_{n,i} \neq Y_{n,i}) = 1$ for $1 \leq i \leq n$. Hence $\sum_{i=1}^{n} \phi(X_{n,i}, Y_{n,i}) = 0$ a.s..
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Example 2

- Let $X_1, \ldots, X_n$ be iid with continuous df F, mean μ_1 and variance σ_1^2.
- Let $Y_1, \ldots, Y_n$ be iid with continuous df G, mean μ_2 and variance σ_2^2.
- Suppose that $G(x) = F(\frac{x-a}{b})$.
- Then, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_{n:i} Y_{n:i} \xrightarrow{a.s.} \mu_1 \mu_2 + \sigma_1 \sigma_2.$$

- Apply above to X_i's and $-Y_i$'s, then, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_{n:i} Y_{n:n-i+1} \xrightarrow{a.s.} \mu_1 \mu_2 - \sigma_1 \sigma_2.$$

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Example 3 Let (X_i, Y_i) be independent bivariate normal with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation coefficient ρ.

- From Theorem 1, as $n \to \infty$,

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$$\frac{\sum_{i=1}^{n} X_{n:i} Y_{n:i} - n(\mu_1\mu_2 + \sigma_1\sigma_2)}{\sqrt{n}} \xrightarrow{dist} N(0, \sigma^2)$$

where

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Thank you for your attention !