Asymptotic Expansion with Double Layers of Markov Chain Forward Equations

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Inroduction and Basic Results

Let X_t be an (possibly) inhomogeneous Markov Chain on finite state space $\{$1,2,...,m $\}$ with transition rate

$$Q(t) = (q_{(i,j)}(t))_{1 \leq i,j \leq m}.$$

We have

$$\Sigma_j q_{(i,j)}(t) = 0, \quad q_{(i,j)} \geq 0 \text{ if } i \neq j.$$

Let $p_{i,j}(t) = p(X_t = j | X_0 = i)$.
Then the backward equation is the following

$$\begin{aligned} \partial p / \partial t &= Q(t) p(t) \\ p_{(i,j)}(0) &= \delta_{i,j} \end{aligned}$$

And the forward equation is

$$\begin{aligned} \partial p/\partial t &= p(t)Q(t) \\ p_{(i,j)}(0) &= \delta_{i,j} \end{aligned}$$

Now let the transition rate matrix be

$$Q^{\epsilon}(t) = 1/\epsilon \cdot Q_1(t) + Q_0(t)$$

and consider the following perturbed forward equation

$$\begin{aligned} \partial p^{\epsilon}/\partial t &= p^{\epsilon}(t)Q^{\epsilon}(t) \\ p^{\epsilon}(0) &= p_0 \end{aligned}$$

- If Q_1 is irreducible, then let $Q_0 = 0$.
- If Q_1 has many irreducible classes, Q_0 is taken to be a non-trivial chain.

The results from [1] is that there are functions

- $f^{(0)}, f^{(1)}(t), ..., f^{(n)}(t)$ (regular part) and
- $h^{(0)}, h^{(1)}(t), ..., h^{(n)}(t)$ (boundary layer) such that
- $sup_{0\leq t\leq T}|p^{\epsilon}(t) - (\Sigma_{i=1}^{n}\epsilon^{i}f^{(i)}(t) + \Sigma_{i=1}^{n}\epsilon^{i}h^{(i)}(t/\epsilon))| = O(\epsilon^{n+1})$.
- Moreover, we have that $h^{(i)}(\tau) \leq Kexp(-\tau\gamma)$ for some positive constants τ and γ for each $h^{(i)}$.

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The problem we are interested in is that the transition rate matrix now is

- $$Q^{\epsilon}(t) = 1/\epsilon^2 \cdot Q_2(t) + 1/\epsilon \cdot Q_1(t) + Q_0(t).$$

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- $$sup_{0\le t\le T} \quad |p^{\epsilon} - (\Sigma_{i=1}^{n}\epsilon^{i}f^{(i)}(t) + \Sigma_{i=1}^{n}\epsilon^{i}h^{(i)}(t/\epsilon) + \Sigma_{i=1}^{n}\epsilon^{i}g^{(i)}(t/\epsilon^{2}))| = O(\epsilon^{n+1})$$

  Moreover, we want to have that
- $h^{(i)}(\tau) \le Kexp(-\tau\gamma)$ and $g^{(i)}(\tau) \le Kexp(-\tau\gamma)$ for some positive constants τ and γ for each $h^{(i)}$ and $g^{(i)}$.
- The most sigificant term $Q_2(t)$ is assumed to have several weakly irreducible classes.

We first consider the toy example where $Q^{\epsilon}(t) = 1/\epsilon \cdot Q$ and $m = 2$.

- $$Q = \begin{pmatrix} -r & r \\ u & -u \end{pmatrix}$$

In this simple case, we can solve $p^{\epsilon}(t)$ explicitely as follows.

- $$\begin{aligned} p^{\epsilon}(t) &= p_o \cdot exp(tQ/\epsilon) = v + p_0 \cdot exp(tQ/\epsilon) - v \\ &= v + (p_0 - v) \cdot exp(tQ/\epsilon) + v \cdot exp(tQ/\epsilon) - v \\ &= v + (p_0 - v) \cdot (exp(tQ/\epsilon) - V) + (p_0 - v) \cdot V \\ &= v + (p_0 - v) \cdot (exp(tQ/\epsilon) - V) \end{aligned}$$

  Here, v is the invariant distribution of Q and $V = 1_{mx1} \cdot v$.
- Hence, $f^{(0)} = v, h^{(0)} = (p_0 - v) \cdot (exp(tQ/\epsilon) - V)$ and $f^{(i)}(t) = h^{(i)} = 0, i \geq 1$.

The second example is also easy. Let $Q^\epsilon = 1/\epsilon \cdot Q_1 + Q_0$ where

$$Q_1 = \begin{pmatrix} -r_1(t) & r_1(t) & 0 & 0 \\ u_1(t) & -u_1(t) & 0 & 0 \\ 0 & 0 & -r_1(t) & r_1(t) \\ 0 & 0 & u_1(t) & -u_1(t) \end{pmatrix}$$

and

$$Q_0 = \begin{pmatrix} -r_2(t) & 0 & r_2(t) & 0 \\ 0 & -r_2(t) & 0 & r_2(t) \\ u_2(t) & 0 & u_2(t) & 0 \\ 0 & u_2(t) & 0 & u_2(t) \end{pmatrix}$$

It is analytically solvable and we only display $f^{(0)}$.

We have that

$$f^{(0)}(t) = (v_1(t)\theta^{(1)}, v_1(t)\theta^{(2)}))$$

where $v_1(t) = (u_1(t)/(r_1 + u_1), u_1(t)/(r_1 + u_1))$.
The functions $\theta^{(1)}(t), \theta^{(2)}(t)$ are two scalars and represent the weiths of the two recurrent classes respectively. They satisfy the following equation

$$d(\theta^1, \theta^2)/dt = (\theta^1, \theta^2) \cdot \begin{pmatrix} -r_2 & r_2 \\ u_2 & -u_2 \end{pmatrix}$$

with the initial condition

$$(\theta^1(0), \theta^2(0)) = (P^0_1 + P^0_2, P^0_3 + P^0_4).$$

One non-trivial example is that $Q^{\epsilon} = 1/\epsilon^2 \cdot Q_2 + 1/\epsilon^1 \cdot Q_1 + Q_0$ where

$$Q_2 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

and

$$Q_0 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

This model is also solvable analyticaly and we have

- $$\begin{aligned} p_1(t) &= 1/2(p_1^0 + p_2^0 - 1/2)exp(-t(2+2\epsilon/\epsilon)) \\ &+1/2(p_1^0 + p_3^0 - 1/2)exp(-t(2+2\epsilon^2/\epsilon^2)) \\ &+1/2(p_1^0 + p_4^0 - 1/2)exp(-t(2+2\epsilon/\epsilon^2)) + 1/4 \end{aligned}$$

- $$\begin{aligned} p_2(t) &= 1/2(p_1^0 + p_2^0 - 1/2)exp(-t(2+2\epsilon/\epsilon)) \\ &-1/2(p_1^0 + p_3^0 - 1/2)exp(-t(2+2\epsilon^2/\epsilon^2)) \\ &-1/2(p_1^0 + p_4^0 - 1/2)exp(-t(2+2\epsilon/\epsilon^2)) + 1/4 \end{aligned}$$

- $$\begin{aligned} p_3(t) &= -1/2(p_1^0 + p_2^0 - 1/2)exp(-t(2+2\epsilon/\epsilon)) \\ &+1/2(p_1^0 + p_3^0 - 1/2)exp(-t(2+2\epsilon^2/\epsilon^2)) \\ &-1/2(p_1^0 + p_4^0 - 1/2)exp(-t(2+2\epsilon/\epsilon^2)) + 1/4 \end{aligned}$$
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One can easily find the corresponding $f^{(i)}$, $h^{(i)}$ and $g^{(i)}$ from $p_i(t)$.

We want to solve the following equation :

$$\begin{aligned} dP^\epsilon/dt &= P^\epsilon \cdot (1/\epsilon^2 \cdot Q_2(t) + 1/\epsilon Q_1(t) + Q_0(t)) \\ P^\epsilon(0) &= p^0 \end{aligned}$$

And we want to find functions $f^{(i)}$, $h^{(i)}$ and $g^{(i)}$ such that

$$sup_{0\le t\le T}|P^\epsilon(t) - f^{n,\epsilon}(t) - h^{n,\epsilon}(t/\epsilon) - g^{n,\epsilon}(t/\epsilon^2)| = O(\epsilon^{n+1})$$

where

- $f^{n,\epsilon} = \Sigma_{i=0}^{n}\epsilon^i f^{(i)}(t)$,
- $h^{n,\epsilon} = \Sigma_{i=0}^{n}\epsilon^i h^{(i)}(t/\epsilon)$ and
- $g^{n,\epsilon} = \Sigma_{i=0}^{n}\epsilon^i g^{(i)}(t/\epsilon^2)$.
- Moreover, $h^{(i)}$ and $g^{(i)}$ have exponential decay for each i.

Matched Expansion
regular part

Let $L^{\epsilon}p = \epsilon^2 \partial p/\partial t - p \cdot (Q_2(t) + \epsilon Q_1(t) + \epsilon^2 Q_0(t))$ and first consider the regular part.

- $L^{\epsilon}(\Sigma \epsilon^i f^{(i)}(t)) = 0$
- For the ϵ^0 and ϵ^1 terms, we have $f^{(0)} \cdot Q_2 = 0$ and $f^{(1)} \cdot Q_2(t) + f^{(0)} \cdot Q_1(t) = 0$.
- For ϵ^2 and in general ϵ^n terms, we have $f^{(2)}(t) \cdot Q_2(t) = \partial f^{(0)}/\partial t - f^1 \cdot Q_1 - f^{(0)} \cdot Q_0$ and $f^{(n)}(t) \cdot Q_2(t) = \partial f^{(n-2)}/\partial t - f^{(n-1)}(t) \cdot Q_1(t) - f^{(n-2)} Q_0(t)$.

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For the first layer, we have
$\epsilon \cdot d/d\tau_1(\Sigma\epsilon^i h^{(i)}(\tau_1)) = (\Sigma\epsilon^i h^{(i)}(\tau_1))(Q_2(t)+\epsilon Q_1(t)+\epsilon^2 Q_0(t))$
where $\tau_1 = t/\epsilon$.

We next expand
$Q_2(t) = \Sigma_{k=0}^{n} d^k Q_2(0)/d^k t(\epsilon\tau_1)^k/k! + R^{n+1}(\epsilon\tau_1)$, $Q_1(t) = \Sigma_{k=0}^{n} d^k Q_1(0)/d^k t(\epsilon\tau_1)^k/k! + R^{n+1}(\epsilon\tau_1)$ and
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- $\epsilon^0, 0 = h^{(0)}(\tau) \cdot Q_2(0)$.
- $\epsilon^1, \partial h^{(0)}/\partial\tau = h^{(0)}(\tau_1)(Q_2^1(0)\tau_1 + Q_1(0)) + h^{(1)} \cdot Q_2(0)$.

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- $\epsilon^0, 0 = h^{(0)}(\tau) \cdot Q_2(0)$.
- $\epsilon^1, \partial h^{(0)}/\partial\tau = h^{(0)}(\tau_1)(Q_2^1(0)\tau_1 + Q_1(0)) + h^{(1)} \cdot Q_2(0)$.

For terms of ϵ^2 and higher, we have

(ϵ^2 terms)

$$\begin{aligned}\partial h^{(1)}/\partial\tau_1 &= h^{(0)}(Q_2^2(0)\tau_1^2/2 + Q_1^1(0)\tau_1 + Q_0)\\ &+h^{(1)}(Q_2^1(0)\tau_1 + Q_1(0))\\ &+h^{(2)}(Q_2(0))\end{aligned}$$

(for ϵ^n terms)

$$\begin{aligned}\partial h^{(n-1)}/\partial\tau_1 &= h^{(0)}(Q_2^n(0)/n!\tau_1^n + Q_1^{n-1}(0)/(n-1)!\tau_1^{n-1}\\ &+Q_0^{n-2}(0)/(n-2)!\tau_1^{n-2})\\ \\ &+h^{(1)}(Q_2^{n-1}(0)/(n-1)!\tau_1^{n-1}\\ &+Q_1^{n-2}(0)/(n-2)!\tau_1^{n-2}\\ &+Q_0^{n-3}(0)/(n-3)!\tau_1^{n-3})\\ \\ &+\ldots\\ &+h^{(n)}\cdot Q_2(0).\end{aligned}$$

For the second layer, we have

$$d/d\tau_2(\Sigma\epsilon^i g^{(i)}(\tau_2)) = (\Sigma\epsilon^i g^{(i)}(\tau_2))(Q_2(t) + \epsilon Q_1(t) + Q_0(t))$$

where $\tau_2 = t/\epsilon^2$. We also expand $Q_2(t) = \Sigma_{k=0}^{n} d^k Q_2(t)/dt^k(0)(\epsilon^2\tau_2)^k/k! + R^{n+1}(\epsilon^2\tau_2)$, $Q_1(t) = \Sigma_{k=0}^{n} d^k Q_1(t)/dt^k(0)(\epsilon^2\tau_2)^k/k! + R^{n+1}(\epsilon^2\tau_2)$ and $Q_0(t) = \Sigma_{k=0}^{n} d^k Q_0(t)/dt^k(0)(\epsilon^2\tau_2)^k/k! + R^{n+1}(\epsilon^2\tau_2)$. Here R^{n+1} are the remainder terms.

- For ϵ^0 terms, we have

$$dg^{(0)}(\tau_2)/d\tau_2 = g^{(0)}(\tau_2) \cdot Q_2(0).$$

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For $n = 2k$, we have

$$\begin{aligned} dg^{(2k)}/d\tau_2 &= g^{(2k)}(\tau_2)\cdot Q_2(0) \\ &+\Sigma_{j=1}^{k} g^{(2k-2j)}(\tau_2)\cdot(\tau_2^j/j!d^jQ_2(0)/dt^j \\ &+\tau_2^{j-1}/(j-1)!d^{j-1}Q_0(0)/dt^{j-1}) \\ &+\Sigma_{j=0}^{k-1} g^{(2k-(2j+1))}(\tau_2)\cdot\tau_2^j/j!d^jQ_1(0)/dt^j. \end{aligned}$$

For $n = 2k+1$, we have

$$\begin{aligned} dg^{(2k+1)}/d\tau_2 &= g^{(2k+1)}(\tau_2)\cdot Q_2(0) \\ &+\Sigma_{j=1}^{k} g^{(2k+1-2j)}(\tau_2)\cdot(\tau_2^j/j!d^jQ_2(0)/dt^j \\ &+\tau_2^{j-1}/(j-1)!d^{j-1}Q_0(0)/dt^{j-1}) \\ &+\Sigma_{j=0}^{k-1} g^{(2k-2j)}(\tau_2)\cdot\tau_2^j/j!d^jQ_1(0)/dt^j. \end{aligned}$$

And finally,

$$dg^{(n)}/d\tau_2 = g^{(n)}\cdot Q_2(0) + r^{(n)}$$

where $r^{(n)}$ is the remainder.

To solve the above equations, we need initial conditions.
From the expansion,

$$sup_{0\le t\le T}|P^{\epsilon}(t)-f^{n,\epsilon}(t)-h^{n,\epsilon}(t/\epsilon)-g^{n,\epsilon}(t/\epsilon^2)|=O(\epsilon^{n+1})$$

we have at $t=0$,
$p^{\epsilon}(0)=p_0=\Sigma_i\epsilon^i f^{(i)}(0)+\Sigma_i\epsilon^i h^{(i)}(0)+\Sigma_i\epsilon^i g^{(i)}(0)$.

- Thus, $f^{(0)}(0)+h^{(0)}(0)+g^{(0)}(0)=p_0$ and
- $f^{(i)}(0)+h^{(i)}(0)+g^{(i)}(0)=0, i\ge 1$.

Multiplying the expansion by 1_{mx1}, we have

- $f^{(0)}(t)\cdot 1_{mx1}=1$ and
- $f^{(i)}(t)\cdot 1_{mx1}=0$ for $i\ge 1$.

Outline

Suppose $Q_2(t)$ has l recurrent classes and let

$$Q_2(t) = \begin{pmatrix} Q_2^1(t) & & & \\ & Q_2^2(t) & & \\ & & \ddots & \\ & & & Q_2^l(t) \end{pmatrix}$$

We write $Q_2(t) = \text{diag}(Q_2^1(t), Q_2^2(t), ..., Q_2^l(t))$ where $Q_2^k(t)$ is an irreducible Markov Chain with m_k states.($\Sigma_{k=1}^l m_k = m$).

Let $\tilde{1} = \text{diag}(1_{m_1}, 1_{m_2}, ..., 1_{m_l})_{mxl}$ where 1_k is a column vector of dimension $kx1$ with all elements 1. Let $V(t) = \text{diag}(v_1(t)), v_2(t), ..., v_l(t))$ where $v_k(t)$ is the row invariant distribution vector of $Q_2^k(t)$:

$$V(t) = \begin{pmatrix} v_1(t) & & & \\ & v_2(t) & & \\ & & \ddots & \\ & & & v_l(t) \end{pmatrix}_{lxm}$$

Let $\bar{Q}_1(t) = V(t) \cdot Q_1(t) \cdot \tilde{1}$. We shall prove the matched expansion under the assumption that $\bar{Q}_1(t)$ is an irreducible Markov Chain with l states. Each state k is now an aggregation of m_k states in the original state space.

To obtain the regular terms $f^{(i)}$, consider first the terms involving ϵ^0.

- $f^{(0)}(t) \cdot Q_2(t) = 0$.
- If $Q_2(t)$ is irreducible, $f^0(t)$ is the quasi-invariant distribution of $Q_2(t)$ and it depends on $Q_2(t)$ only. The ϵ^1 approxmation will depend on $Q_0(t)$ and $Q_1(t)$. Higher order approximation will depend on all the Q's.uju
- Decompose $f^{(0)}(t) = (f^{(0),1}, f^{(0),2}, ..., f^{(0),l})$ where $f^{(0),k}(t) = \vartheta^{(0),k}(t)v^k(t)$ and $\vartheta^{(0),k}(t)$ is a scalar multiplier, $v^k(t)$ is the invariant distribution of $Q_2^k(t)$.
- We have $f^{(0)}(t) \cdot \tilde{1} = \vartheta^0(t)$ and $f^{(0)}(t) = \vartheta^0(t) \cdot V(t)$. (Hence solving $f^{(0)}(t)$ is equivalent to solving $\vartheta^0(t)$).

To solve $\vartheta^0(t)$, we need to consider the next term $f^{(1)}(t)$.

We have $f^{(1)}(t) \cdot Q_2(t) = -f^{(0)}(t) \cdot Q_1(t)$ for the ϵ^1 term.

- Multiply the equation by $\tilde{1}$ and we obtain
- $\vartheta^{(0)}(t) \cdot \bar{Q}_1(t) = 0$. Moreover, $\vartheta^{(0)}(t) \cdot 1_{lx1} = 1$.
- $\vartheta^{(0)}(t)$ is solved uniquely and thus is $f^{(0)}(t)$.
- To obtain $f^{(1)}(t)$, consider the next term in ϵ^2 and get

$$\begin{aligned} f^{(2)}(t) \cdot Q_2(t) &= df^{(0)}(t)/dt - f^{(0)}(t) \cdot Q_0(t) \\ &\quad - f^{(1)}(t) \cdot Q_1(t) \end{aligned}$$

- Multiply the equation by $\tilde{1}$, we have

$$\begin{aligned} 0 &= d\vartheta^{(0)}(t)/dt - \vartheta^{(0)}(t) \cdot V(t) \cdot Q_0(t) \cdot \tilde{1} \\ &\quad - (\vartheta^{(1)}(t) \cdot V(t) + \tilde{b}^0(t)) \cdot Q_1(t) \cdot \tilde{1} \end{aligned}$$

and thus

$$\begin{aligned} \vartheta^{(1)}(t) \cdot \bar{Q}_1(t) &= d\vartheta^{(0)}(t)/dt - \vartheta^{(0)}(t) \cdot V(t) \cdot Q_0(t) \cdot \tilde{1} \\ &\quad - \tilde{b}^0(t) \cdot Q_1(t) \cdot \tilde{1} \end{aligned}$$

In the above, we assume the form $f^{(1)}(t) = \vartheta^{(1)}(t) \cdot V(t) + \tilde{b}^{(0)}(t)$ where $\vartheta^{(1)}(t) \cdot V(t)$ isin the the kernel and $\tilde{b}^{(0)}(t)$ is a particular solution of $f^{(1)}(t) \cdot Q_2(t) = -f^{(0)}(t) \cdot Q_1(t)$ for the ϵ^1.
Moreover, we have $\vartheta^{(1)}(t) \cdot 1_{lx1} = 0$ and

- $f^{(1)}(t)$ is uniquely solved.
- $f^{(i)}(t)$ can then be solved similarly.
- Remark. If $\bar{Q}_1(t)$ is not irreducible, it will then take three equarions to uniquely solve determine f_i. Now it only requires two.

Outline

For the first layer terms, we start from ϵ^0. (This is similar to the regular terms.)

$$h^{(0)}(\tau_1) \cdot Q_2(0) = 0.$$

Similar to $f^{(0)}$, we decompose $h^{(0)}(\tau_1) = \theta^{(0)}(\tau_1) \cdot V(0)$ where $\theta^{(0)}(\tau_1)$ is a $1xl$ row vector and

$$\begin{aligned} h^{(0)}(\tau_1) &= \theta^{(0)}(\tau_1) \cdot V(0) \\ h^{(0)}(\tau_1) \cdot \tilde{1} &= \theta^{(0)}(\tau_1) \quad \text{and} \\ \Sigma_{k=1}^{l} \theta^{(0),k}(\tau_1) &= 0 \end{aligned}$$

Solving $h^{(0)}(\tau_1)$ is equivalent to solving $\theta^{(0)}(\tau_1)$.

Consider ϵ^1 term.

$$\epsilon^1 : dh^{(0)}(\tau_1)/d\tau_1 = h^{(1)}(\tau_1) \cdot Q_2(0) + h^{(0)}(\tau_1) \cdot (\tau_1 dQ_2(0)/dt + Q_1(0)).$$

Multiplying the equation by $\tilde{1}$ and we get

- $d\theta^{(0)}(\tau_1)/d\tau_1 = \theta^{(0)}(\tau_1) \cdot \bar{Q}_1(0)$.
- An initial condition of $\theta^{(0)}(0)$ determines the solution.
- From the expansion $p(0) = f^{(0)}(0) + h^{(0)}(0) + g^{(0)}(0)$, we have $\theta^{(0)}(0) = h^{(0)} \cdot \tilde{1} = p^{(0)} \cdot \tilde{1} - f^{(0)} \cdot \tilde{1}$.
- (Of course, here we need $g^{(0)}(0) \cdot \tilde{1} = 0$)

To see that $g^{(0)}(0) \cdot \tilde{1} = 0$, we need to consider the ϵ^0 term in the expansion of second layer expansion :

$$dg^{(0)}/d\tau_2(\tau_2) = g^{(0)}(\tau_2) \cdot Q_2(0).$$

Multiplying $\tilde{1}$, we have $dg^{(0)}/d\tau_2(\tau_2) \cdot \tilde{1} = 0$ and thus $g^{(0)}(\tau_2) \cdot \tilde{1} = c$. The exponential decay property then implies $g^{(0)}(\tau_2) \cdot \tilde{1} = 0$. Similar methods can be used to obtain higher order terms h^i.

Outline

To find the second layer terms, we first consider ϵ^0.

$$\epsilon^0; dg^{(0)}(\tau_2)/d\tau_2 = g^{(0)}(\tau_2) \cdot Q_2(0).$$

- This differential equation can be uniquely solved if $g^{(0)}(0)$ is given.
- Since $p^{(0)} = f^{(0)}(0) + h^{(0)}(0) + g^{(0)}(0)$ and $f^{(0)}(0), h^{(0)}(0)$ are determined,
- therefore, $g^{(0)} = -f^{(0)}(0) - h^{(0)}(0) + P^{(0)}(0)$ and $g^{(0)}(\tau_2)$ is uniquely solved.
- Similarly, $g^{(i)}$ can be solved.

Outline

To see the exponential decay of $h^{(0)}(\tau_1)$, we consider $h^{(0)}(\tau_1) = \theta^{(0)}(\tau_1) \cdot V(0)$. Since $\theta^{(0)}(\tau_1) = \theta^{(0)}(0) \cdot exp(\tau_1 \bar{Q}_1(0))$ and $\theta^{(0)}(0) = p^0 \cdot \tilde{1} - \vartheta^{(0)}(0)$ is orthogonal to 1_{lx1}, thus

- $$\begin{aligned} \theta^{(0)}(\tau_1) &= \theta^{(0)}(0) \cdot (exp(\tau_1 \bar{Q}_1(0)) - V) + \theta^{(0)}(0) \cdot V \\ &= \theta^{(0)}(0) \cdot (exp(\tau_1 \bar{Q}_1(0)) - V) \end{aligned}$$

  where V is the matrix with each row the invariant distribution of $\bar{Q}_1(0)$.

- The exponential decay of $h(0)(\tau_1)$ now follows because $exp(\tau_1 \bar{Q}_1(0)) - V$ decays exponentially.

We next show the exponential decay of $g(0)(\tau_2)$. Obviously, $g^{(0)}(\tau_2) = g^{(0)}(0) \cdot exp(\tau_2 Q_2(0))$ where $g^{(0)}(0) = p^0 - f^{(0)}(0) - h^{(0)}(0)$. Hence we have

$$\begin{aligned} exp(\tau_2 Q_2(0)) &= \text{diag}(exp(\tau_2 Q_2^1(0)), ..., exp(\tau_2 Q_2^l(0))) \\ &\longrightarrow \text{diag}(1_{m_1 x 1} \cdot v^1(0), ..., 1_{m_l x 1} \cdot v^l(0)) \\ &\text{exponentially as} \quad \tau_2 \rightarrow 0 \end{aligned}$$

where $v^k(0)$ is the invariant distribution of $Q_2^k(0)$.

But $\mathrm{diag}(1_{m_1 x1}\cdot v^1(0),\ldots,1_{m_l x1}\cdot v^l(0)) =$

$$\tilde{1}\cdot \mathrm{diag}(v^1(0),\ldots,v^l(0)) = \begin{pmatrix} 1_{m_1 x1} & & & \\ & 1_{m_2 x1} & & \\ & & \vdots & \\ & & & 1_{m_l x1} \end{pmatrix}\cdot$$

$$\begin{pmatrix} v^1(0) & & & \\ & v^2(0) & & \\ & & \ddots & \\ & & & v^l(0) \end{pmatrix},$$

thus we have

$$g^{(0)}(0) \cdot \mathrm{diag}(1_{m_1 x1} \cdot v^1(0), ..., 1_{m_l x1} \cdot v^l(0)) = 0$$

because $g^{(0)}(0) \cdot \tilde{1} = 0$ which was established earlier. It thus follows that

$$\begin{aligned} g^{(0)}(\tau_2) &= g^{(0)}(0) \cdot exp(\tau_2 Q_2(0)) \\ &= g^{(0)}(0) \cdot (exp(\tau_2 Q_2(0)) \\ &-\mathrm{diag}(1_{m_1 x1} \cdot v^1(0), ..., 1_{m_l x1} \cdot v^l(0))) \\ &\longrightarrow 0 \quad \text{exponentially fast.} \end{aligned}$$

Outline

What remains is to analyze the error. We have the following theorem. Recall that $L^\epsilon = \epsilon^2 df/dt - f \cdot (Q_2 + \epsilon Q_1 + \epsilon^2 Q_0)$. Suppose that

- $sup_{0 \leq t \leq T} |L^\epsilon v^\epsilon(t)| = \eta^\epsilon(t) = O(\epsilon^{k+2})$ and $v^\epsilon(0) = 0$, then we have
- $sup_{0 \leq t \leq T} |v^\epsilon(t)| = O(\epsilon^k)$.
- Th proof hangs on the fact that $v^\epsilon(t) = 1/\epsilon^2 \int_0^t \eta^\epsilon(s) X^\epsilon(t,s) ds$ where $X^\epsilon(t,s)$ is the principal matrix solution of L^ϵ and is bounded.
- THE END.

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- THE END.

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