Renormalization in the model of Brownian motions in Poissonian potentials

Xia Chen

Department of Mathematics
The University of Tennessee, Knoxville
xchen@math.utk.edu

Part of the talk is based on the collaborative works with Alexey Kulik and Jay Rosen.

Our story starts from the book “Brownian motion, obstacles and random media” (A-S. Sznitman), where a Brownian particle moves in a space full of obstacles randomly located in the form of Poissonian cloud.

The central part of the book is to investigate the long term asymptotic behaviors of the trajectory of the particle which survives from being trapped by the obstacles.

The (annealed) Gibbs measure of the form

$$\frac{dQ_t}{dP} = \frac{1}{Z_t} \exp\left\{ -\theta \int_0^t V(B_s)ds \right\}$$

is introduced to generates the trajectory of the Brownian particle surviving from being trapped by the obstacles, where B_s is a d-dim. Brownian motion,

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy) \quad x \in \mathbb{R}^d$$

is called (Piossonian) potential function and $\omega(dy)$ is a independent (of B_s) Poissonian field on $\mathbb{R}^d$ with intensity 1.

In Sznitman's work, the shape function $K(x)$ is assumed to be non-negative bounded and locally supported.

By Newton's law of universal gravitation, on the other hand, some most natural ways to define the potential function are

$$V(x) = \int_{\mathbb{R}^d} \frac{\omega(dy)}{|y-x|^2} \quad \text{and} \quad V(x) = \int_{\mathbb{R}^d} \frac{\omega(dy)}{|y-x|}.$$

The first and second measure, respectively, the net attraction and the net potential at the location x in a gravitational field generated by the Poissonian stars.

Thus, we propose to take

$$K(x) = |x|^{-p}.$$

A disappointing fact is

$$\int_{\mathbb{R}^d} \frac{\omega(dy)}{|y-x|^p} \equiv \infty \quad \forall p > 0$$

which is indicated by the easy computation

$$\mathbb{E} \int_{\mathbb{R}^d} \frac{\omega(dy)}{|y-x|^p} = \int_{\mathbb{R}^d} \frac{dy}{|y-x|^p} = \int_{\mathbb{R}^d} \frac{dy}{|y|^p} = \infty.$$

Renormalization

Let $p > 0$ be fixed and for each N, let $K_N(x) \geq 0$ be bounded and locally supported such that $K_N(x) \uparrow |x|^{-p}$ $(N \to \infty)$. Define the Gibbs measure $Q_{t,N}$ as

$$\frac{dQ_{N,t}}{dP} = \frac{1}{Z_t} \exp\left\{ -\theta \int_0^t V_N(B_s) ds \right\}$$

where

$$V_N(x) = \int_{\mathbb{R}^d} K_N(y-x)\omega(dy) \quad x \in \mathbb{R}^d$$

The key observation is that

$$\mathbb{E}\, V_N(x) = \int_{\mathbb{R}^d} K_N(y-x)dy = \int_{\mathbb{R}^d} K_N(y)dy = \text{constant}_N.$$

Renormalization

Write $\overline{V}_N = V_N(x) - \mathbb{E}\, V_N(x)$. By "renormalization"

$$\frac{dQ_{N,t}}{dP} = \frac{1}{Z_t} \exp\left\{ -\theta \int_0^t \overline{V}_N(B_s)ds \right\}.$$

Notice that

$$\int_0^t \overline{V}_N(B_s)ds = \int_{\mathbb{R}^d} \left[\int_0^t K_N(y - B_s)ds \right] [\omega(dx) - dx]$$

"converges" to

$$\int_{\mathbb{R}^d} \left[\int_0^t \frac{ds}{|x - B_s|^p} \right] [\omega(dx) - dx] = \int_{\mathbb{R}^d} \xi(t,x) [\omega(dx) - dx].$$

Renormalization

When

$$\frac{d}{2} < p < \min\left\{d, \frac{d+2}{2}\right\}$$

the conditional variance

$$\mathrm{Var}\left\{\int_{\mathbb{R}^d} \xi(t,x)\big[\omega(dx) - dx\big]\Big|B\right\} = \int_{\mathbb{R}^d} \xi^2(t,x)dx$$

$$= C\int_0^t\int_0^t \frac{drds}{|B_r - B_s|^{2p-d}} < \infty$$

where the second step follows from the relation

$$\int_{\mathbb{R}^d} \frac{dx}{|x-y|^p|x-z|^p} = \frac{1}{|y-z|^{2p-d}}.$$

Renormalization

In the remaining of the talk, we assume that

$$\frac{d}{2} < p < \min\left\{d, \frac{d+2}{2}\right\}.$$

The Gibbs measure Q_t given by

$$\frac{dQ_t}{dP} = \frac{1}{Z_t} \exp\left\{ -\theta \int_{\mathbb{R}^d} \xi(t, x)\big[\omega(dx) - dx\big] \right\}$$

is well defined and for fixed $t > 0$, $\|Q_{t,N} - Q_t\|_V \to 0$ as $N \to \infty$.

With the Gibbs measure being defined, a natural question is to ask the long term behavior of this model. In particular, it is interesting to see how differently our model behaves from the model in Sznitman's setting.

In this talk, we discuss the long term asymptotics of the partition function Z_t given as

$$\mathbb{E}\,\exp\left\{-\theta\int_{\mathbb{R}^d}\xi(t,x)\Big[\omega(dx)-dx\Big]\right\}$$

with $\theta>0$ being dependent or independent of t.

Brownian motion in Brownian potential

We first consider the model of Brownian motion in Brownian potential, whose trajectory is generated by the Gibbs measure

$$\frac{dQ_t}{dP} = \frac{1}{Z_t} \exp\left\{ -\theta \int_{\mathbb{R}^d} \xi(t,x) W(dx) \right\}$$

where $W(x)$ $(x \in \mathbb{R}^d)$ is a Brownian sheet independent of B_s.

Heuristically,

$$\int_{\mathbb{R}^d} \xi(t,x) W(dx) = \int_0^t U(B_s) ds$$

with the (Brownian) potential function

$$U(x) = \int_{\mathbb{R}^d} \frac{W(dy)}{|y-x|^p} \qquad x \in \mathbb{R}^d.$$

Brownian motion in Brownian potential

This model is concerned with a Brownian particle (carrying one unit electronic charge with fixed sign) moving in an electronic field, where $W(x)$ symbols the spatial distribution of a cloud of electronic charges with random $\pm$-signs..

According to Coulomb's law, $U(x)$ represents the net force and the net potential, when $p = 2$ and $p = 1$, respectively, at the location x in the field.

Brownian motion in Brownian potential

The mathematical difference between gravitational and the electronic fields can be substantial due to the fact one has sign flapping and another doesn't. On the other hand, the renormalization in Poissonian case may link two models together.

Theorem (Chen-Kulik (2010))

$$t^{-\frac{d+4-2p}{2}} \int_{\mathbb{R}^d} \xi(t,x)\left[\omega(dx) - dx\right] \xrightarrow{d} \int_{\mathbb{R}^d} \xi(1,x) W(dx)$$

Brownian motion in Brownian potential

Theorem (Chen-Rosen (2010))

$$\lim_{t\to\infty} t^{-\frac{d+4-2p}{d+2-2p}} \log \mathbb{E} \exp\left\{\theta \int_{\mathbb{R}^d} \xi(t,x)W(dx)\right\}$$

$$= \Lambda \theta^{\frac{4}{d+2-2p}} \left(\frac{\pi^{d/2}}{2} \frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)}\right)^{\frac{2}{d+2-2p}}$$

where

$$\Lambda = \sup_{g\in\mathcal{F}_d}\left\{\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^{2p-d}}dxdy - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\}$$

and $\mathcal{F}_d = \left\{g \in \mathcal{L}^2(\mathbb{R}^d);\ \ \|g\|_2 = 1,\ \nabla g \in {}^2(\mathbb{R}^d)\right\}$.

Super-critical case

The setting of Poissonian potential is far more complicated due to lack of self-similarity. There are three regimes according to the scale of $\theta = \theta(t)$.

Theorem (Chen-Kulik (2010))

$$\lim_{t\to\infty} \frac{1}{b_t} \log \mathbb{E} \exp\left\{ -\frac{b_t^{p/d}}{t}\theta \int_{\mathbb{R}^d} \xi(t,x)\Big[\omega(dx) - dx\Big] \right\}$$
$$= \theta^{d/p} \int_{\mathbb{R}^d} \Big[\exp\{-|x|^{-p}\} - 1 + |x|^{-p}\Big] dx$$

for any $\theta > 0$ *and positive* b_t *with* $b_t / t^{\frac{d}{d+2}} \to \infty$.

Super-critical case

Remark 1. Taking $b_t = t^{d/p}$ gives

$$\lim_{t\to\infty} t^{-d/p} \log \mathbb{E} \exp\left\{ -\theta \int_{\mathbb{R}^d} \xi(t,x)\Big[\omega(dx) - dx\Big]\right\}$$
$$= \theta^{d/p} \int_{\mathbb{R}^d} \Big[\exp\{-|x|^{-p}\} - 1 + |x|^{-p}\Big] dx$$

Notice that

$$\frac{d}{p} < \frac{d+4-2p}{d+2-2p}$$

We see a substantial difference between Poisonian and Brownian settings.

Super-critical case

Remark 2. Recall (Donsker-Varadhan, Sznitman) that when the shape function $K(x)$ is bounded and locally supported,

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E} \exp\left\{ -\theta \int_0^t V(B_s)ds \right\} = -\frac{d+2}{2}\omega_d^{\frac{2}{d+2}}\left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}}$$

where ω_d is the volume of the d-dimensional unit ball, and λ_d is the principal eigenvalue of $(1/2)\Delta$ on the d-dimensional unit ball with zero boundary values.

In particular, we observe a shape-insensitivity in above result.

Our result is drastically different from the one for the case of local shape.

Critical case

Theorem (Chen-Kulik (2010))

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E} \exp\left\{ -t^{-\frac{d+2-p}{d+2}}\theta \int_{\mathbb{R}^d} \xi(t,x)\big[\omega(dx)-dx\big]\right\}$$
$$= \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2 dx\right\}$$

where $\varphi(a) = e^{-a} - 1 + a$ *(*$a \geq 0$*).*

Remark. At the deviation scale $t^{d/(d+2)}$,

$$t^{-\frac{d+2-p}{d+2}}\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big] \approx C\int_{\mathbb{R}^d} V(B_s)ds$$

Sub-critical case

Theorem (Chen-Kulik (2010))

$$\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{-b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\theta\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big]\right\}$$
$$=\Lambda\theta^{\frac{4}{d+2-2p}}\left(\frac{\pi^{d/2}}{2}\frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)}\right)^{\frac{2}{d+2-2p}}$$

for b_t *satisfying* $b_t\to\infty$, $b_t=o\left(t^{d/(d+2)}\right)$.

Remark. For small scale b_t, Poisonian potential meets Brownian potential.

Sketch of the proof

Proof of

$$\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\exp\left\{-\frac{b_t^{p/d}}{t}\theta\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big]\right\}$$
$$=\theta^{d/p}\int_{\mathbb{R}^d}\Big[\exp\{-|x|^{-p}\}-1+|x|^{-p}\Big]dx$$

under $b_t/t^{\frac{d}{d+2}}\to\infty$.

Sketch of the proof

By Fubini's theorem

$$\mathbb{E}\ \exp\Big\{-\frac{b_t^{p/d}}{t}\theta\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big]\Big\}$$
$$=\mathbb{E}\ \exp\Big\{\int_{\mathbb{R}^d}\varphi\Big(\frac{b_t^{p/d}}{t}\theta\xi(t,x)\Big)dx\Big\}$$

where $\varphi(a)=e^{-a}-1+a$ ($a\in\mathbb{R}^+$) is a non-negative convex function.

Sketch of the proof

Recall that

$$\xi(t,x) = \int_0^t \frac{ds}{|x - B_s|^p}$$

By Jensen inequality,

$$\int_{\mathbb{R}^d} \varphi\Big(\frac{b_t^{p/d}}{t}\theta\xi(t,x)\Big)dx \le \frac{1}{t}\int_0^t \Big[\int_{\mathbb{R}^d} \varphi\Big(\frac{\theta b_t^{p/d}}{|x - B_s|^p}\Big)dx\Big]ds$$

$$= \int_{\mathbb{R}^d} \varphi\Big(\frac{\theta b_t^{p/d}}{|x|^p}\Big)dx = \theta^{d/p} b_t \int_{\mathbb{R}^d} \varphi\Big(\frac{1}{|x|^p}\Big)$$

This leads to the desired upper bound.

Sketch of the proof

On the other hand, let $\epsilon > 0$ be fixed. On the event $\{\sup_{s\le t}|B_s| \le t^{1/(d+2)}\}$,

$$\xi(t,x) = \int_0^t \frac{ds}{|x-B_s|^p} \sim \int_0^t \frac{ds}{|x|^p} = t|x|^{-p}$$

uniformly for all x satisfying $|x| \ge \epsilon\theta^{1/p} b_t^{1/d}$ as $t \to \infty$

Sketch of the proof

Hence

$$\mathbb{E}\exp\left\{\int_{\mathbb{R}^d}\varphi\Big(\frac{b_t^{p/d}}{t}\theta\xi(t,x)\Big)dx\right\}$$

$$\geq \exp\left\{\int_{\{|x|\geq\epsilon\theta^{1/p}b_t^{1/d}\}}\varphi\Big(\frac{\theta b_t^{p/d}}{|x|^p}\Big)dx\right\}\mathbb{P}\Big\{\sup_{s\leq t}|B_s|\leq t^{1/(d+2)}\Big\}$$

$$=\exp\left\{\theta^{d/p}b_t\int_{\{|x|\geq\epsilon\}}\varphi\Big(\frac{1}{|x|^p}\Big)dx\right\}\mathbb{P}\Big\{\sup_{s\leq t}|B_s|\leq t^{1/(d+2)}\Big\}$$

Finally, the lower bound follows from the classic small ball estimate

$$\mathbb{P}\Big\{\sup_{s\leq t}|B_s|\leq t^{1/(d+2)}\Big\}\geq \exp\Big\{-Ct^{\frac{d}{d+2}}\Big\}=\exp\big\{o(b_t)\big\}$$

Sketch of the proof

Proof of

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E} \exp\left\{ -t^{-\frac{d+2-p}{d+2}}\theta \int_{\mathbb{R}^d} \xi(t,x)\big[\omega(dx)-dx\big]\right\}$$
$$= \sup_{g\in\mathcal{F}_d}\left\{ \int_{\mathbb{R}^d}\varphi\left(\theta\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

where we recall that $\varphi(a) = e^{-a} - 1 + a$

Sketch of the proof

$$\begin{aligned}
&\mathbb{E}\,\exp\left\{-t^{-\frac{d+2-p}{d+2}}\theta\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big]\right\}\\
&=\mathbb{E}\,\exp\left\{\int_{\mathbb{R}^d}\varphi\Big(t^{-\frac{d+2-p}{d+2}}\theta\xi(t,x)\Big)dx\right\}\\
&=\mathbb{E}\,\exp\left\{t^{\frac{d}{d+2}}\int_{\mathbb{R}^d}\varphi\Big(\theta t^{-\frac{d}{d+2}}\xi(t^{\frac{d}{d+2}},x)\Big)dx\right\}
\end{aligned}$$

Thus, the desired conclusion follows from the following Donsker-Varadhan type of theorem with t being replaced by $t^{d/(d+2)}$.

Sketch of the proof

Theorem (Chen-Kulik (2010))

Let $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be a non-decreasing, differentiable and convex function such that* $\psi(0) = 0$ *and that*

$$\int_0^1 a^{-\frac{d+p}{p}} \psi(a)da < \infty$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ t \int_{\mathbb{R}^d} \psi\Big(\frac{1}{t}\xi(t,x)\Big)dx \right\}$$
$$= \sup_{g\in\mathcal{F}_d} \left\{ \int_{\mathbb{R}^d} \psi\bigg(\int_{\mathbb{R}^d} \frac{g^2(y)}{|x-y|^p}dy \bigg)dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}$$

Sketch of the proof

Proof of

$$\lim_{t\to\infty} t^{-\frac{d+4-2p}{d+2-2p}} \log \mathbb{E} \exp\left\{\theta \int_{\mathbb{R}^d} \xi(t,x)W(dx)\right\}$$
$$= \Lambda \theta^{\frac{4}{d+2-2p}} \left(\frac{\pi^{d/2}}{2} \frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)} \right)^{\frac{2}{d+2-2p}}$$

where

$$\Lambda = \sup_{g\in\mathcal{F}_d} \left\{ \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^{2p-d}} dxdy - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}$$

Sketch of the proof

By Fubini's theorem,

$$\mathbb{E}\ \mathrm{exp}\left\{\theta\int_{\mathbb{R}^d}\xi(t,x)W(dx)\right\}=\mathbb{E}\ \mathrm{exp}\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d}\xi^2(t,x)dx\right\}$$

Taking $\psi(a)=(\theta^2/2)a^2$ in the previous theorem leads to

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\ \mathrm{exp}\left\{\frac{\theta^2}{2t}\int_{\mathbb{R}^d}\xi^2(t,x)dx\right\}$$

$$=\sup_{g\in\mathcal{F}_d}\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{g^2(y)}{|y-x|^p}dy\right]^2dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

$$=\Lambda\theta^{\frac{4}{d+2-2p}}\left(\frac{\pi^{d/2}}{2}\frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)}\right)^{\frac{2}{d+2-2p}}$$

Sketch of the proof

By scaling,

$$\mathbb{E}\exp\left\{\frac{\theta^2}{2t}\int_{\mathbb{R}^d}\xi^2(t,x)dx\right\} = \mathbb{E}\exp\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d}\xi^2\left(t^{\frac{d+2-2p}{d+4-2p}},\ x\right)dx\right\}$$

Replacing $t^{\frac{d+2-2p}{d+4-2p}}$ by t gives

$$\lim_{t\to\infty} t^{-\frac{d+4-2p}{d+2-2p}}\log\mathbb{E}\exp\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d}\xi^2(t,x)dx\right\}$$
$$= \Lambda\theta^{\frac{4}{d+2-2p}}\left(\frac{\pi^{d/2}}{2}\frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)}\right)^{\frac{2}{d+2-2p}}$$

This completes the proof.

Sketch of the proof

Proof of

$$\lim_{t\to\infty}\frac{1}{b_t}\log \mathbb{E}\exp\left\{-b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\theta\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big]\right\}$$

$$=\Lambda\theta^{\frac{4}{d+2-2p}}\left(\frac{\pi^{d/2}}{2}\frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)}\right)^{\frac{2}{d+2-2p}}$$

under $b_t = o\left(t^{d/(d+2)}\right)$.

Sketch of the proof

By Fubini,

$$\mathbb{E}\exp\left\{-b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\theta\int_{\mathbb{R}^d}\xi(t,x)\big[\omega(dx)-dx\big]\right\}$$
$$=\mathbb{E}\exp\left\{\int_{\mathbb{R}^d}\varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\xi(t,x)\right)dx\right\}$$

Here we recall $\varphi(a)=e^{-a}-1+a$. By the fact $\varphi(a)\le(1/2)a^2$,

$$\mathbb{E}\exp\left\{\int_{\mathbb{R}^d}\varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\xi(t,x)\right)dx\right\}$$
$$\le\mathbb{E}\exp\left\{\frac{\theta^2}{2}b_t^{-1}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{2}}\int_{\mathbb{R}^d}\xi^2(t,x)dx\right\}$$

Sketch of the proof

By scaling right hand side is equal to

$$\mathbb{E}\ \exp\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d}\xi^2\left(b_t^{\frac{d+2-2p}{d+4-2p}},\ x\right)dx\right\}$$

Replacing t by $b_t^{\frac{d+2-2p}{d+4-2p}}$ in the LDP for $\psi(a)=(\theta^2/2)a^2$,

$$\limsup_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\ \exp\left\{\int_{\mathbb{R}^d}\varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\xi(t,x)\right)dx\right\}$$

$$\leq\Lambda\theta^{\frac{4}{d+2-2p}}\left(\frac{\pi^{d/2}}{2}\frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)}\right)^{\frac{2}{d+2-2p}}$$

This gives the upper bound.

Sketch of the proof

On the other hand,

$$\int_{\mathbb{R}^d} \varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\xi(t,x)\right)dx$$
$$\overset{d}{=} \left(\frac{t}{b_t}\right)^{d/2}\int_{\mathbb{R}^d}\varphi\left(\left(b_t t^{-\frac{d}{d+2}}\right)^{\frac{d+2}{4}}\theta b_t^{-1}\xi(b_t,x)\right)dx$$

It is straightforward to verify that for any > 0, the function $q(c) = c^{-2}\varphi(ca)$ is decreasing on $\mathbb{R}^+$. Noticing that $b_t t^{-\frac{d}{d+2}} \to 0$,

$$\varphi\left(\left(b_t t^{-\frac{d}{d+2}}\right)^{\frac{d+2}{4}}\theta b_t^{-1}\xi(b_t,x)\right) \geq \left(b_t t^{-\frac{d}{d+2}}\right)^{\frac{d+2}{2}}\left(\frac{\theta}{\epsilon}\right)^2\varphi\left(\epsilon b_t^{-1}\xi(b_t,x)\right)$$

for any fixed $\epsilon > 0$ and large t.

Sketch of the proof

Summarizing what we have,

$$\mathbb{E}\,\exp\left\{\int_{\mathbb{R}^d}\varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\xi(t,x)\right)dx\right\}$$
$$\geq \mathbb{E}\,\exp\left\{b_t\left(\frac{\theta}{\epsilon}\right)^2\int_{\mathbb{R}^d}\varphi\left(\epsilon b_t^{-1}\xi(b_t,x)\right)dx\right\}$$

Using our general theorem with t being replaced by b_t and with $\psi_\epsilon(a)=\left(\frac{\theta}{\epsilon}\right)^2\varphi\left(\epsilon a\right)$,

$$\liminf_{t\to\infty}\frac{1}{b_t}\log\mathbb{E}\,\exp\left\{\int_{\mathbb{R}^d}\varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}}\xi(t,x)\right)dx\right\}$$
$$\geq \sup_{g\in\mathcal{F}_d}\left\{\theta^2\int_{\mathbb{R}^d}\psi_\epsilon\left(\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

Sketch of the proof

Notice $\psi_\epsilon(a) \uparrow (1/2)a^2$ as $\epsilon \to 0^+$. Letting $\epsilon \to 0^+$ on the right hand side

$$\liminf_{t\to\infty} \frac{1}{b_t} \log \mathbb{E} \exp\left\{ \int_{\mathbb{R}^d} \varphi\left(\theta b_t^{-1/2}\left(\frac{b_t}{t}\right)^{\frac{d+4-2p}{4}} \xi(t,x)\right) dx \right\}$$

$$\geq \sup_{g\in\mathcal{F}_d} \left\{ \frac{\theta^2}{2} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{g^2(y)}{|x-y|^p} dy \right]^2 dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}$$

$$= \Lambda \theta^{\frac{4}{d+2-2p}} \left(\frac{\pi^{d/2}}{2} \frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)} \right)^{\frac{2}{d+2-2p}}$$

That gives the lower bound.

Thank you!