Large Deviations for Non-ergodic Markov processes

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Since Donsker-Varadhan's fundamental works on large deviations for Markov processes from 1975-1983, most large deviation results for Markov processes are obtained under certain (strong) ergodicity conditions, or for process with special initial distributions. There were a few works handling non-ergodic processes. But there are still many interesting problems to be studied. In this work we will mainly consider continuous time absorbing and related Markov processes. The motivation came from the study of Interacting Particle Systems(IPS).

1.Large deviations for absorbing Markov processes

Let $E = E_0 \cup E_1$ be a Polish space, $X = \{X_t,\ t \geq 0\}$ be a homogeneous Feller Markov process on some probability space $(\Omega,\ \mathcal{F},\ \mathcal{P})$, with state space E. E_0 will be interpreted as an absorbing set, and E be a class of irreducible transient states.Define

$$\tau = \inf\{t \geq 0,\ X_t \in E_0\}$$

to be the absorbing time. τ can also be interpreted as the exit time from E_1. We will consider large deviations for the empirical measures defined by

$$L_t = \frac{1}{t} \int_0^t \delta_{X_s} ds,\ t \geq 0$$

.

1.Large deviations for absorbing Markov processes

For each $t \geq 0$, L_t takes its values in $M_1(E)$, the space of probability measures on E, equipped with the weak topology, and the Borel σ-algebra. We have the following

Theorem 1.1. *Under the above notations, if E_1 is such that 1_{E_1} is continuous on E, and X is irreducible (in certain sense) on E_1, then for each $x \in E$, $\{P_x(L_t \in \cdot),\ t \geq 0\}$ satisfies a weak large deviation principle, i.e.,*

1.Large deviations for absorbing Markov processes

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x(L_t \in K,\ \tau > t)$$

$$\leq -\inf\{I(\mu),\ \mu \in K,\ \mu(E_1) = 1\} \qquad (1.1)$$

for each compact $K \subset M_1(E)$ *and*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x(L_t \in G,\ \tau > t)$$

$$\geq -\inf\{I(\mu),\ \mu \in G,\ \mu(E_1) = 1\} \qquad (1.2)$$

for each open $G \subset M_1(E)$.

where I is Donsker-Varadhan's I functional defined by

$$I(\mu) = - \inf_{u \in C_b(E), u \geq 1} \int \frac{Lu}{u} d\mu,$$

and L is the generator of the process. If in addition, the family $\{P_x(L_t \in \cdot), \ t \geq 0\}$ is exponentially tight, the we have a full LDP, i.e., (1.1) holds for every closed $F \subset M_1(E)$. In particular,

$$\lim_{t \to \infty} \frac{1}{t} \log P_x(\tau > t) = - \inf_{\mu(E_1)=1} I(\mu).$$

We are more interested in the following

Corollary 1.2. *Under the conditions of Theorem 1.1, including the exponential tightness,* $\{P_x(L_t \in \cdot | \tau > t)\ t \geq 0\}$ *satisfies a full LDP with the rate function*

$$I_\tau(\mu) \equiv I(\mu) - \inf_{\mu(E_1)=1} I(\mu)$$

if $\mu(E_1) = 1$ *(i.e.,* $\mu \in M_1(E_1)$*);* $\equiv +\infty$ *otherwise.*

2.Connection with quasi-stationary distribution

$I(\mu) = 0$ iff μ is invariant for X. How about I_τ? If $I_\tau(\mu) = 0$, is μ a quasi-stationary distribution of the MP w.r.t E_0, i.e., is

$$P_\mu(X_t \in \cdot | \tau > t)$$

independent of t? The answer is: not necessarily. The zeros of I_τ are related to "Quasi-mean stationary distribution" of the MP, i.e., such $\mu \in M_1(E_1)$ which is a weak limit of the form

$$\int f d\mu = \lim_{t \to \infty} E_\nu [L_t(f) | \tau > t]$$

for some initial distribution ν. Now if we consider a Markov chain X with a countable state space $E = E_0 \cup E_1$, with E_0 and E_1 the same interpretation as before. Denote by

$$P_{i,j}(t) = P(X_t = j | X_0 = i)$$

for $i,j \in E$. Then we have the following

2.Connection with quasi-stationary distribution

(Theorem 2.1) *If the Mrkov chain is λ-positive for some $\lambda > 0$, i.e.,*

$$\lim_{t\to\infty} P_{i,j}(t)e^{\lambda t} > 0$$

for all $i, j \in E_1$, then the only μ such that $I_\tau(\mu) = 0$ is quasi-mean stationary.

3.An example from interacting particle systems

As an example, we consider the contact process $X = \{X_t,\ t \geq 0\}$ on $E = \{0,\ 1\}^{\Lambda}$.
(1) If $\Lambda \subset Z^d$ is a finite set, X is a continuous time Markov chain with the finite state space E. $E_0 = \vec{0}$, the identically 0 state, is the only absorbing state. Thus Theorem can be applied. In particular, we can sow that

$$\inf_{t>0} \frac{\log P(\tau > t)}{t} = \lim_{t\to\infty} \frac{\log P(\tau > t)}{t} = - \inf_{\mu(E_1)=1} I(\mu) < 0.$$

from these we can prove that X is λ-positive for some $\lambda > 0$, thus X has a unique quasi-mean stationary distribution.

(2)If $\Lambda = Z^d$, and we start the process from a state x with $\sum_{i \in Z^d} x(i) < \infty$, then the process is a continuous time Markov chain with countable and infinite state space. In this case, as Theorem 1.1 shows, the large deviation lower bounds hold. Thus similar to the case (1), X is λ-positive for some $\lambda > 0$, thus X has a unique quasi-mean stationary distribution.

Thanks You!