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Collisions of Random Walks

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1. The Question

A simple random walk $\{X_n\}$ in Z^d

$$X_n = \xi_1 + \xi_2 + \cdots + \xi_n.$$

where $P(\xi = e_i) = 1/(2d)$.

a different view: $X_n = X_{n-1} + \xi_n$.

X at x chooses a neighbor of x with equal probability.

A random walk on graph G is a Markov chain that jumps from one vertex to a neighbor with equal probability.

Run two independent random walks X and Y on G.

$N(\omega) = \{n; X_n(\omega) = Y_n(\omega)\}$.

Question: $N \geq 1$? $N < \infty$?

2. Some Backgrounds

Polya wandered around in a park near Zurich when he noticed that he kept encountering the same couple.

In $\boldsymbol{Z}^{\boldsymbol{d}}$, consider $\boldsymbol{Z_n = X_n - Y_n}$

Transience/recurrence,

Monotone property: If $\boldsymbol{G_1} \subset \boldsymbol{G_2}$, and if $\boldsymbol{G_2}$ is recurrent, then $\boldsymbol{G_1}$ is also recurrent.

Polya wandered around in a park near Zurich when he noticed that he kept encountering the same couple.

In Z^d, consider $Z_n = X_n - Y_n$

Transience/recurrence, (recurrent graph)

Monotone property: If $G_1 \subset G_2$, and if G_2 is recurrent, then G_1 is also recurrent.

0-1 Law: If G is recurrent, and $X_0 = Y_0$, then $P(N = \infty) = 1$ or 0.

derived by a theorem of S. Orey in early 1960.

1) $\boldsymbol{P(N = \infty) = 1}$,

2) $\boldsymbol{EN < \infty}$,

3) $\boldsymbol{P(N < \infty) = 1, EN = \infty}$.

1) $\boldsymbol{P(N=\infty)=1, (\Longrightarrow EN=\infty)}$

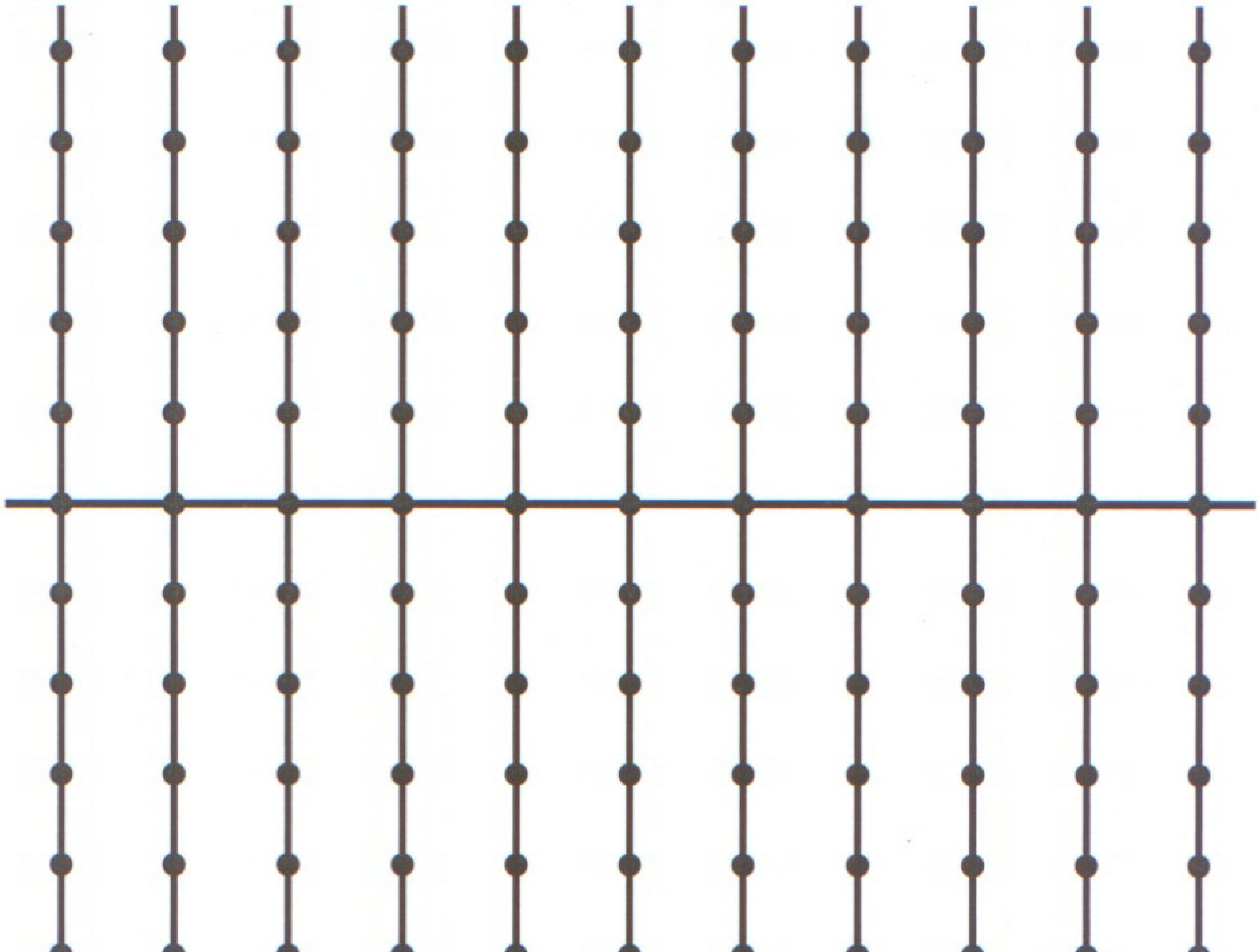
2) $\boldsymbol{EN<\infty, (\Longrightarrow P(N<\infty)=1)}$

3) $\boldsymbol{P(N<\infty)=1, EN=\infty}$.

voter process $\longleftrightarrow$ coalescing random walks.

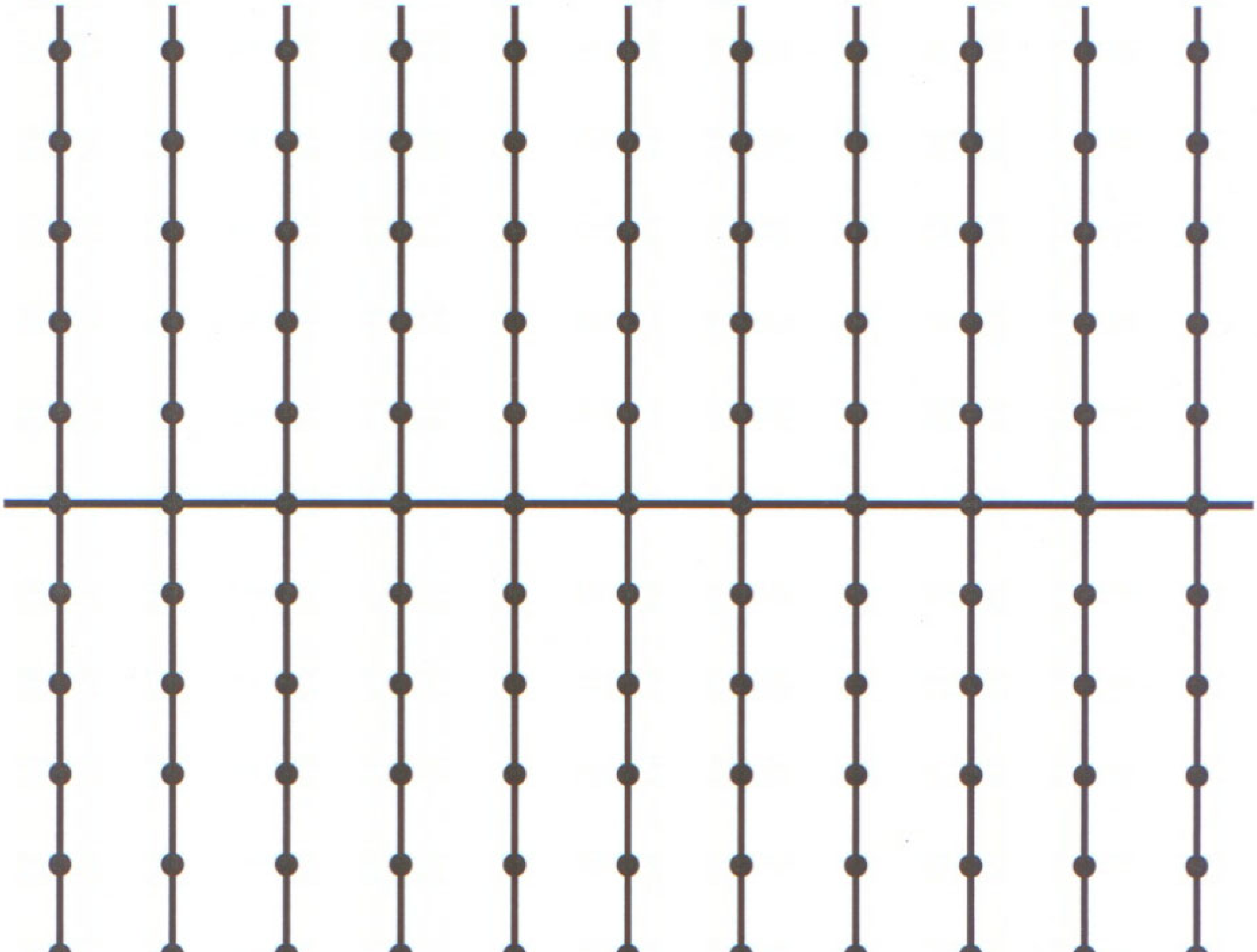
Liggett first found an example of case 3).

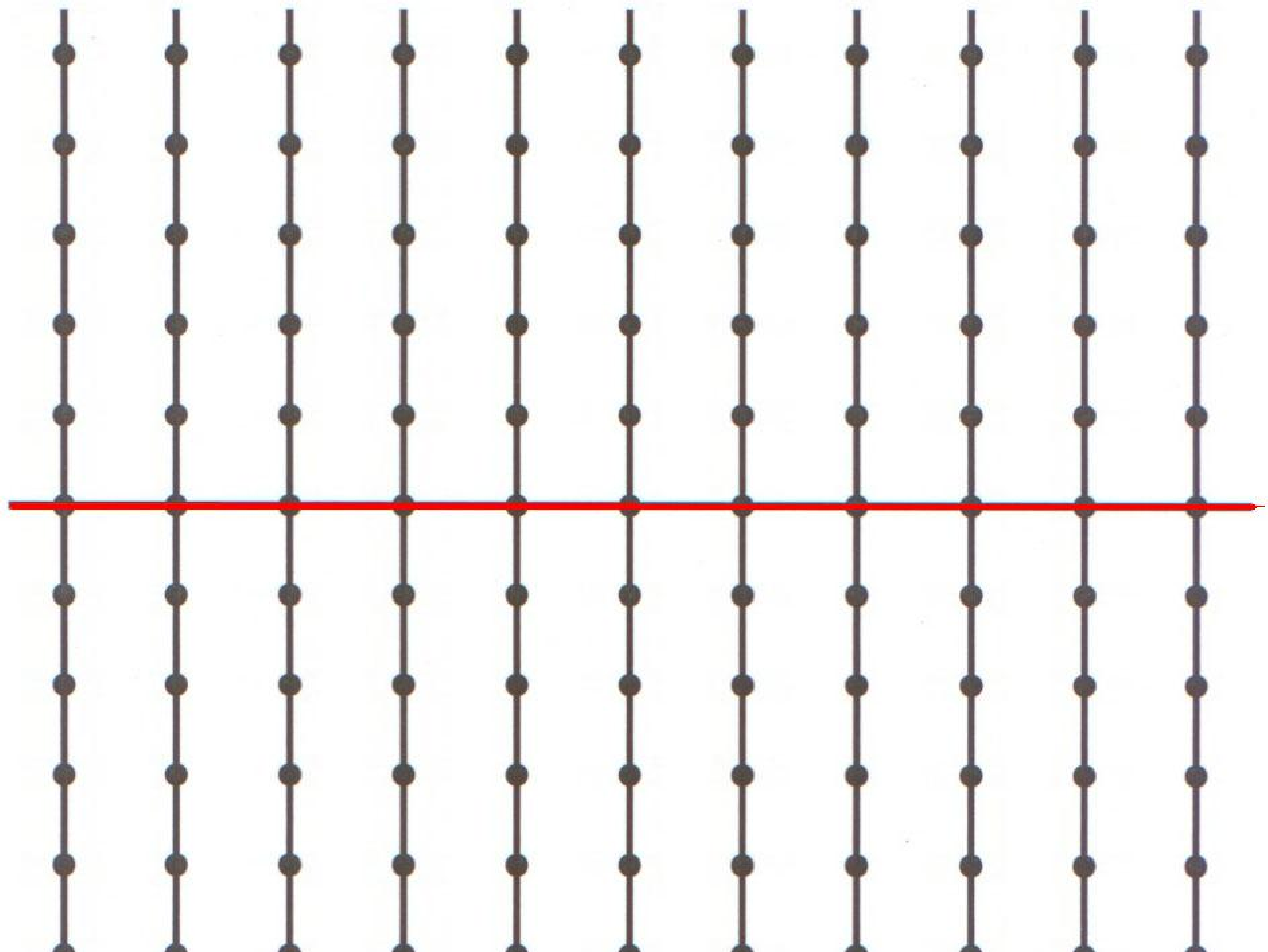
An easy example, Comb($\mathbb{Z}$), by Krishnapur and Peres (2004)

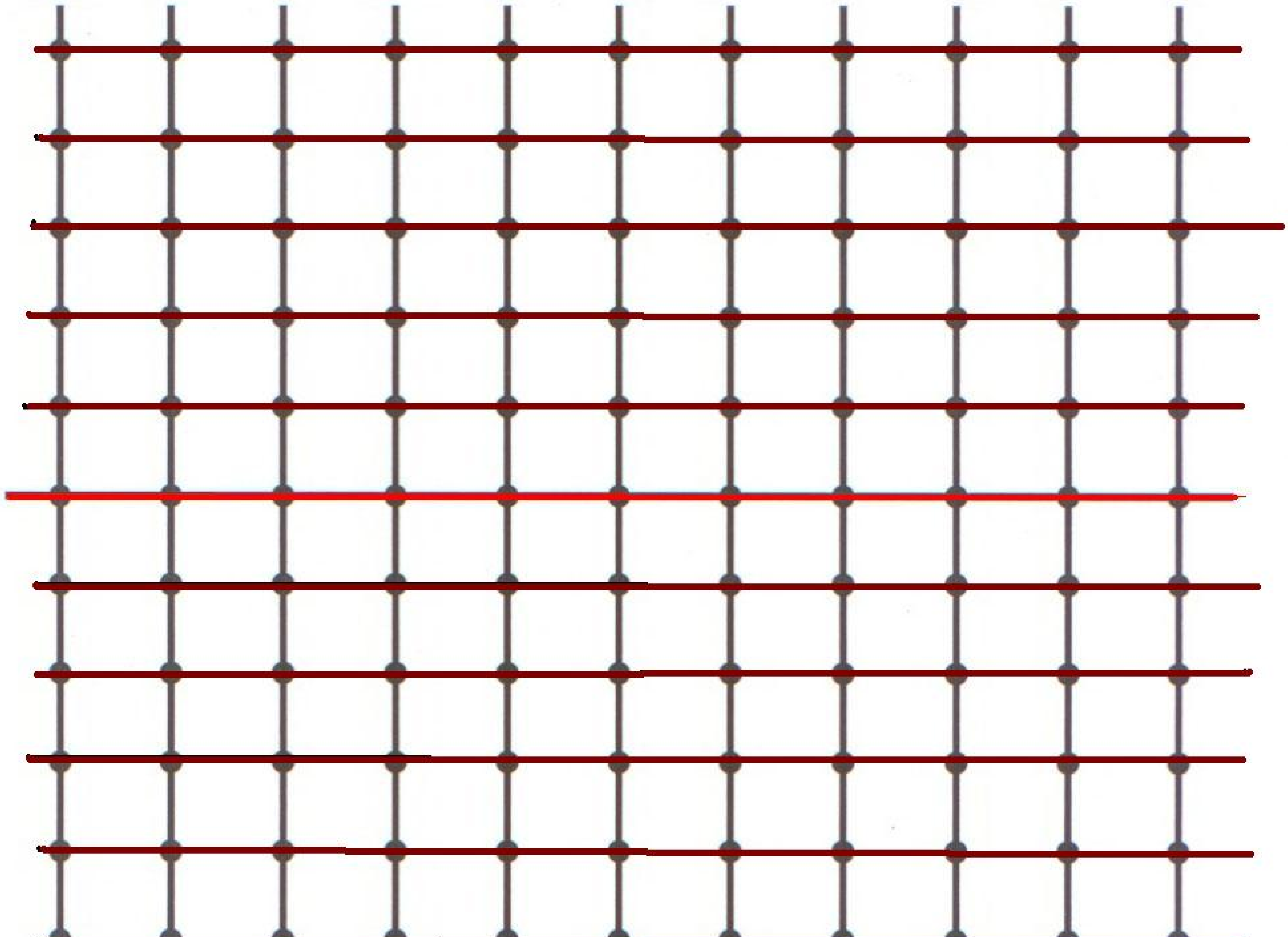


Intuitively, the excursions on the attached $\mathbb{Z}$ cost too much time for two walkers to meet.

$\boldsymbol{Z}^{\mathbf{1}} \subset \mathrm{Comb}(\boldsymbol{Z}) \subset \boldsymbol{Z}^{\mathbf{2}}$.

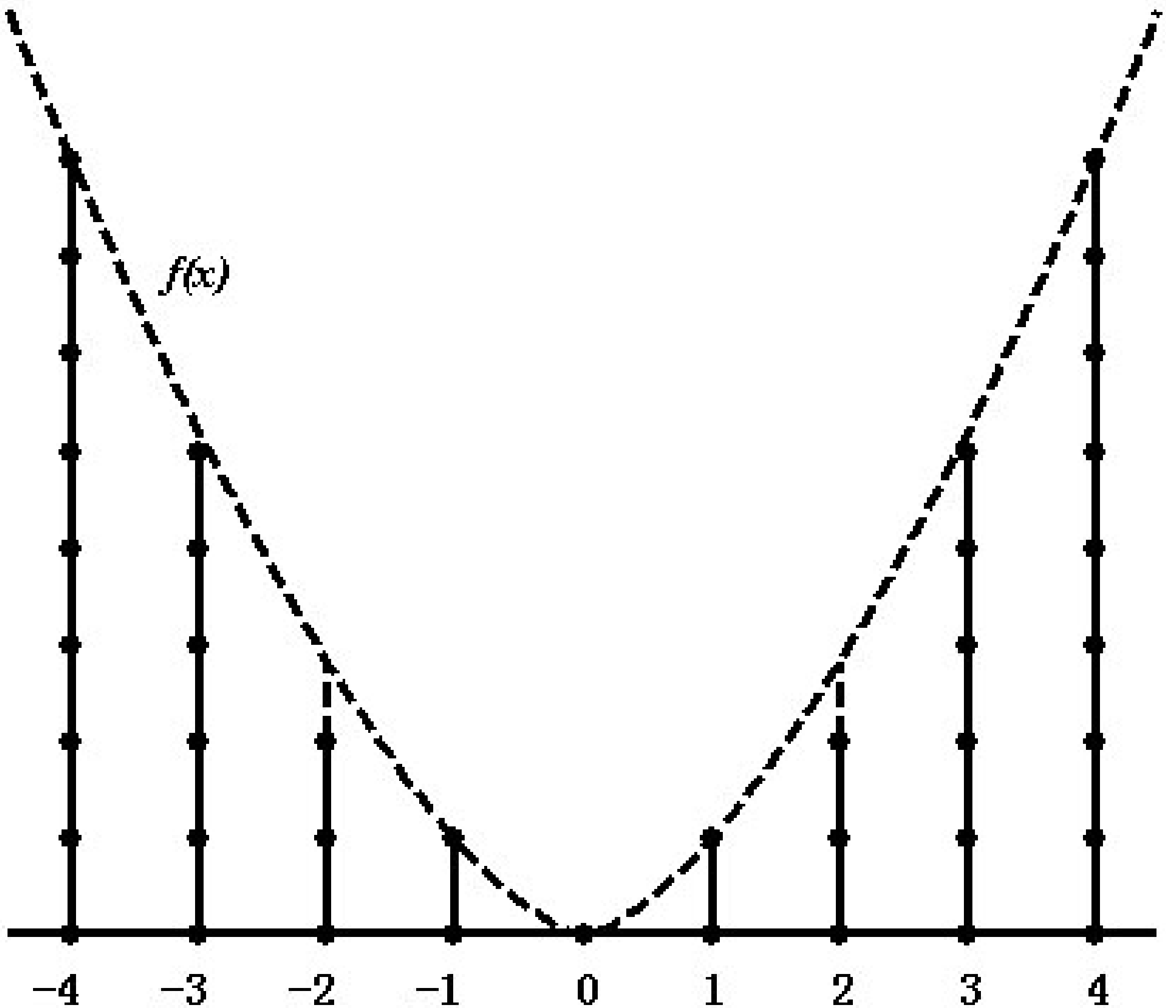






Monotonicity fails.

Can the property of monotonicity be true in a more restrictive setting?



f(x)
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Presumably, a phase transition is expected to occur:

$N = \infty$ *a.s.* if $f(x)$ increases slowly in x; and

$N < \infty$ *a.s.* otherwise.

Theorem 0: Let G be a wedge comb with profile f. Suppose $f(x) \leq |x|^{\alpha}$ for some $\alpha < 1/5$, whenever $|x|$ is large. Then $N = \infty$ *a.s.*.

i.e., two independent simple random walks on G with continuous time parameter will meet infinitely often.

Chen. D., Wei, B. and Zhang, F., A note on the finite collision property of random walks. *Statistics and Probability Letters*, **78**, 1742-1747, (2008). first reported in Changsha in 2006.

3. New results

Theorem 1. Let G be a wedge comb with profile $f = |x|^{\alpha}$. Then $N = \infty$ *a.s.* if $\alpha \leq 1$ and $N < \infty$ *a.s.* if $\alpha > 1$.

Namely, a phase transition occurs!

Proved by Martin Barlow, Yuval Peres and Perla Sousi (2009).

Tom Mountford also addressed this problem.

Theorem 2. Let G be a wedge comb with profile f. Define

$$\bar{f}(n) = \max\{1, f(n), f(n-1), \cdots, f(-n)\}.$$

If $\sum 1/\bar{f}(n) = \infty$, then $N = \infty$ *a.s.*.

Example: $f(n) = |n| \log |n|$.

If $f(n) = |n|(\log |n|)^{\alpha}$ and $\alpha > 2$, then $N < \infty$ *a.s.*.

Proved by Xinxing Chen.

We believe that an open cluster of the Bernoulli bond percolation on G should resemble the original graph G.

Theorem 3. Consider $\mathbb{Z}^2$ and let $p > 1/2$. There exists $\Omega_0 \subseteq \Omega$ with $\mathbf{P}_p(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $x \in \mathcal{C}_\infty(\omega)$. Then $N = \infty$ *a.s.*.

X. Chen & D. Chen, Two random walks on the open cluster of Z^2 meet infinitely often, To appear in *Science China Mathematics*, 2010.

It was also independently proved by Martin Barlow, Yuval Peres and Perla Sousi.

Bernoulli bond percolation. Each edge e is assigned a Bernoulli r.v. η_e, with $\mathrm{P}_p(\eta_e = 1) = p \in [0, 1]$. Edges e with $\eta_e = 1$ are called open.

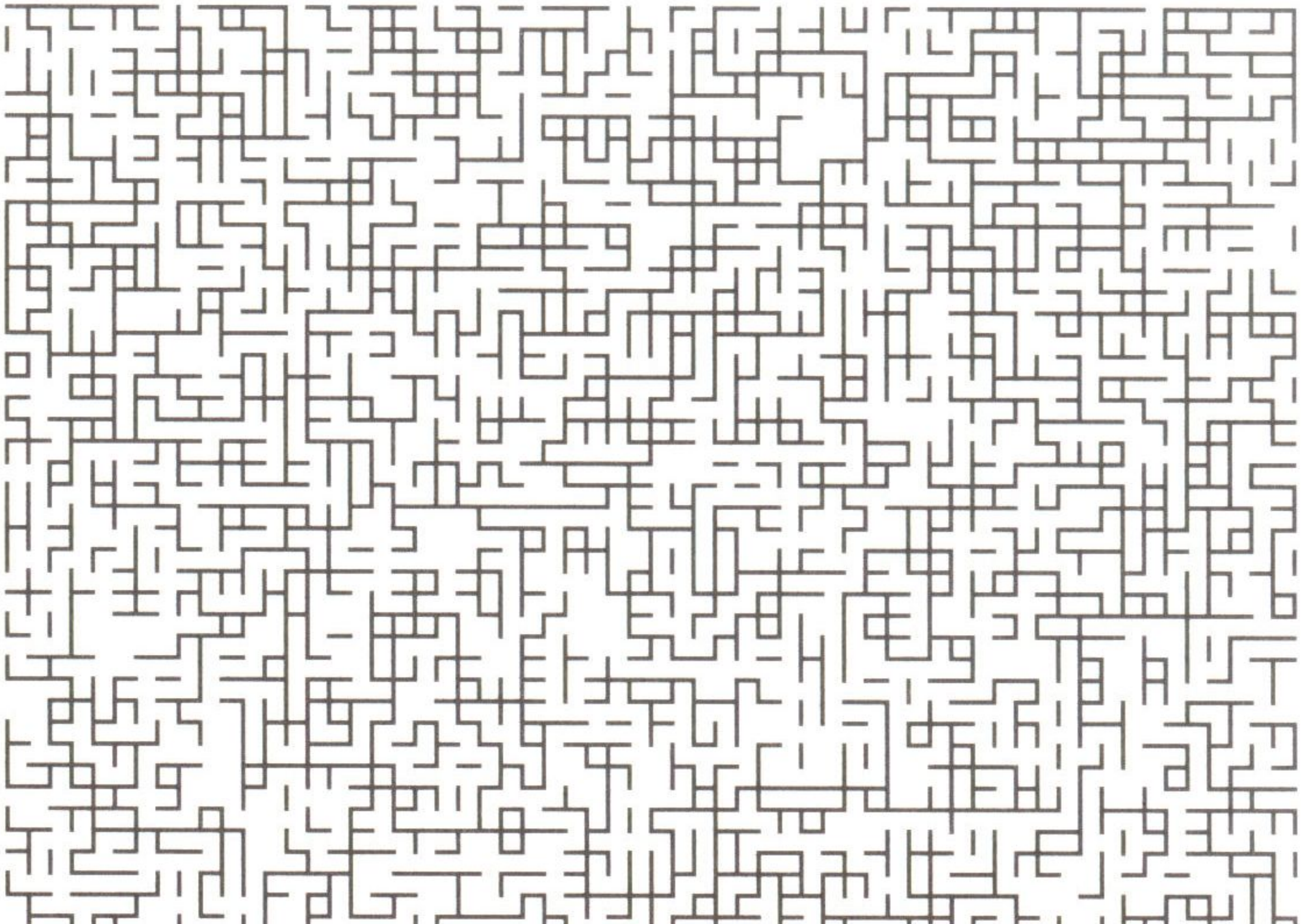
An open cluster is a (connected) component of the random graph consisting of all vertices and all open edges.

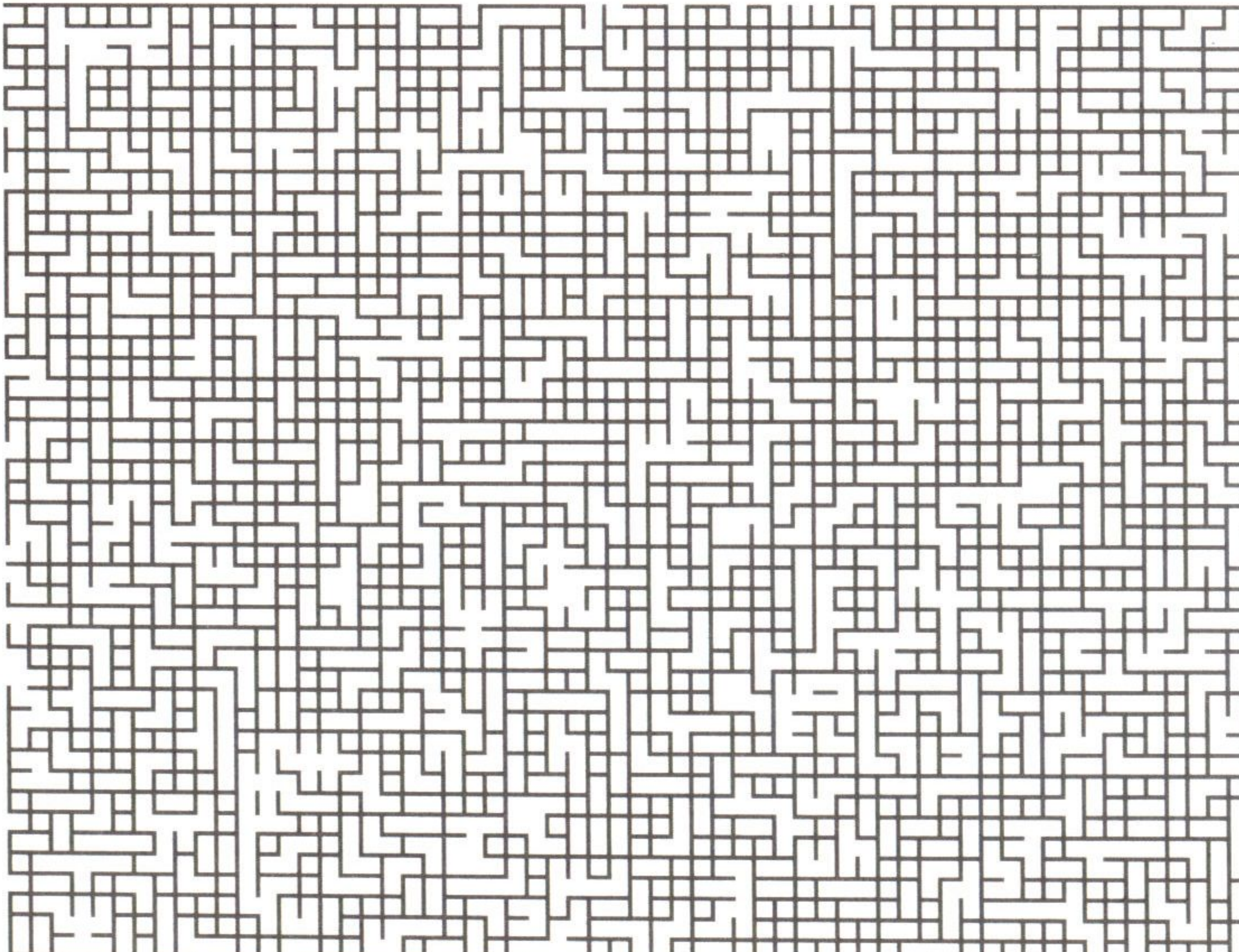
$\mathcal{C}(x)$ is the open cluster that contain x, i.e. the set of vertices y such that x and y are connected by an open path.

The size of $\mathcal{C}(x)$ is increasing in p. Define the critical probability

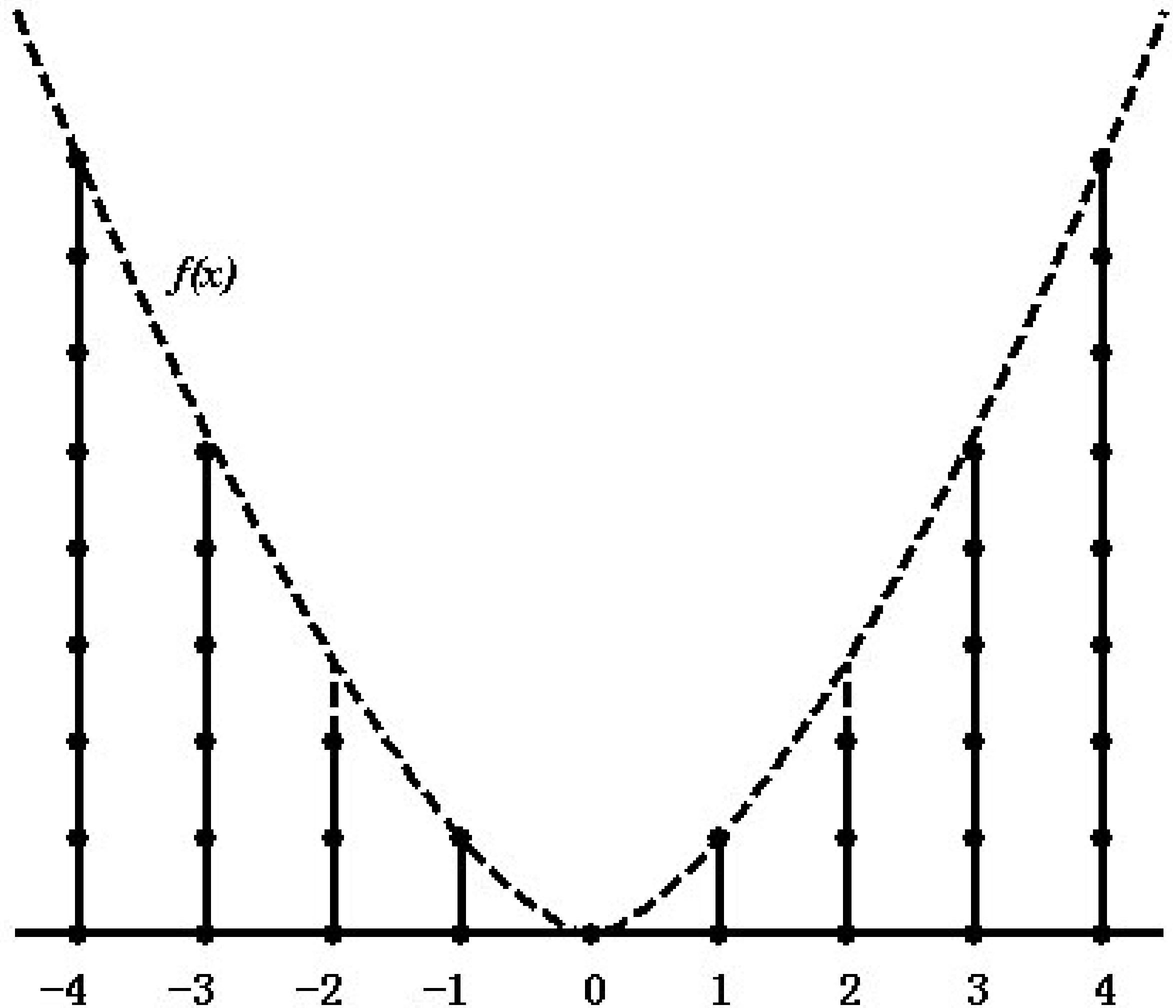
$$p_c = \inf\{p; \mathrm{P}_p(|\mathcal{C}(x)| = \infty) > 0\}.$$

e.g. Z^2, $p_c = 1/2$.





4. Elements of the proof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Decompose the random walk X_n on the comb wedge as (U_n, V_n), horizontal and vertical motions.

Let $\tau_{m+1} = \inf\{n > \tau_m, U_n \neq U_{\tau_m}\}$. Then $\{U_{\tau_m}\}$ is a simple random walk in Z^1.

We have a good estimate of size of $\{n \leq N, U_n = U_n'\}$.

We then further estimate that $P(n \leq N, U_n = U_n', V_n = V_n')$ by the moment method.

If H is a non-negative r.v., then $P(H > 0) \geq (EH)^2/EH^2$.

Lemma: Let $\omega \in \Omega_0$ and $x, y \in \tilde{C}_\infty(\omega)$. Let $X = (X_t)$ be a continuous time simple random walk starting from x on $\mathcal{C}_\infty(\omega)$, $Y = (Y_t)$ a continuous time simple random walk starting from y. If X and Y are independent, then

$$\mathrm{P}(X_t = Y_t \text{ for some } t > 1\ ) \geq \delta,$$

where δ is a strictly positive constant and dependent on p at most.

The proof of Theorem 3 is based on this lemma, which in turn is based on the heat kernel estimate of the simple random walk on the infinite cluster, due to Martin Balow, Random walks on supercritical percolation clusters, *Annals of Probability*, **8** 3024-3084, (2004).

Lemma (Balow) Let $p > p_c(\mathbb{Z}^d)$. There exists $\Omega_1 \subseteq \Omega$ with $\mathrm{P}_p(\Omega_1) = 1$ and r.v. V and $S_x, x \in \mathbb{Z}^d$, such that $V(\omega) < \infty$ and $S_x(\omega) < \infty$ for each $\omega \in \Omega_1, x \in \mathcal{C}_\infty(\omega)$. There exist constants c_i which is strictly positive and dependent only on p, such that for all $n \geq V(\omega)$,

$$|\mathcal{C}_\infty(\omega) \cap [-n, n]^2| \geq c_1 n^2; \tag{1}$$

and for all $x, y \in \mathcal{C}_\infty(\omega), t \geq 1$ with

$$S_x(\omega) \vee |x - y|_\infty \leq t. \tag{2}$$

the transition density $q_t^\omega(x, y)$ of a continuous time simple random walk satisfies

$$\begin{aligned} c_2 t^{-d/2} \exp\{-c_3|x-y|^2_\infty/t\} &\leq q^\omega_t(x,y) \\ \leq c_4 t^{-d/2} \exp\{-c_5|x-y|^2_\infty/t). \end{aligned}$$

Moreover, the tail of the random variable S_x satisfies

$$\mathrm{P}_p(x \in \mathcal{C}_\infty, S_x \geq n) \leq c_6 \exp\{-c_7 n^{c_8}\}. \tag{3}$$

Additionally, if $|x-y|_\infty > t$ then

$$q^\omega_t(x,y) \leq c_9 \exp\left\{-c_{10}|x-y|_\infty(1+\log\frac{|x-y|_\infty}{t})\right\}. \tag{4}$$

5. Discussions

Time parameter: discrete vs continuous

A collision is well defined for the discrete-time Markov chain.

For continuous-time Markov chains (Chung Process?), we say $X_t = Y_t$ infinitely often if there is an infinite sequence $\{t_1, s_1, t_2, s_2, \cdots\}$ such that $t_1 < s_1 < t_2 < s_2 < \cdots$, $X_{t_i} = Y_{t_i}$ and $X_{s_i} \neq Y_{s_i}$ for all $i \geq 1$.

Transience/recurrence is independent of time parameter.

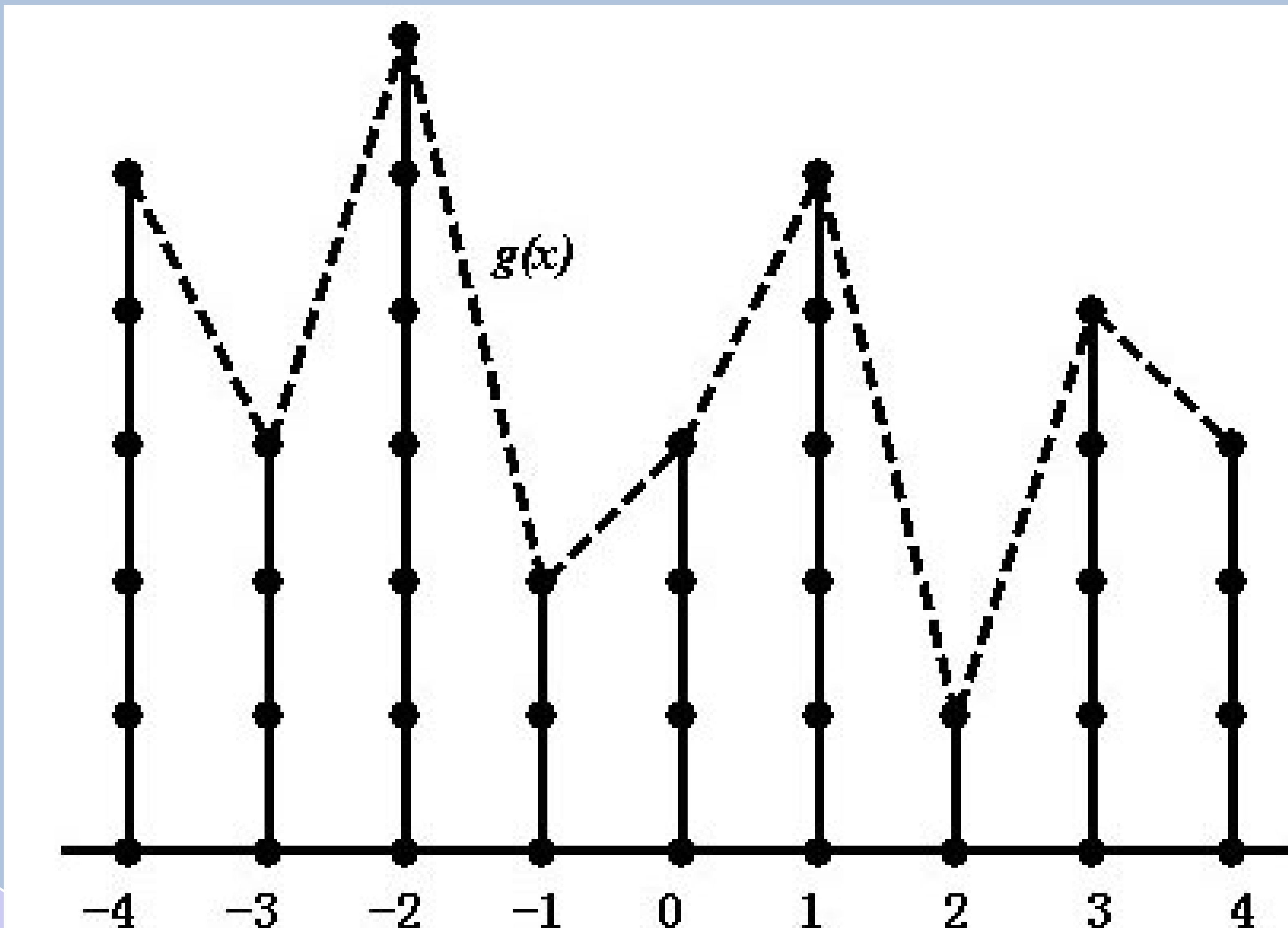
The finite/infinity collision property is also independent of time parameters if the graph is quasi-transitive.

We believe this is also true if the degree of the graph is bounded.

In general we DO NOT KNOW, and can only verify it case by case.

Three random walks

random environments



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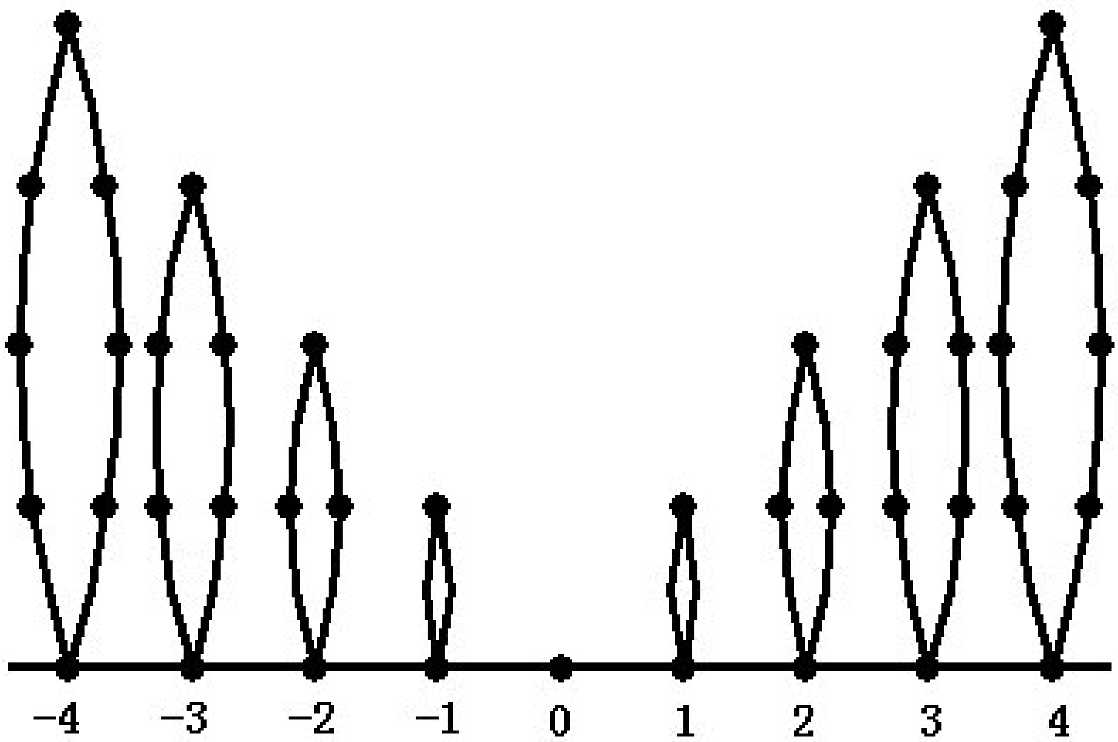
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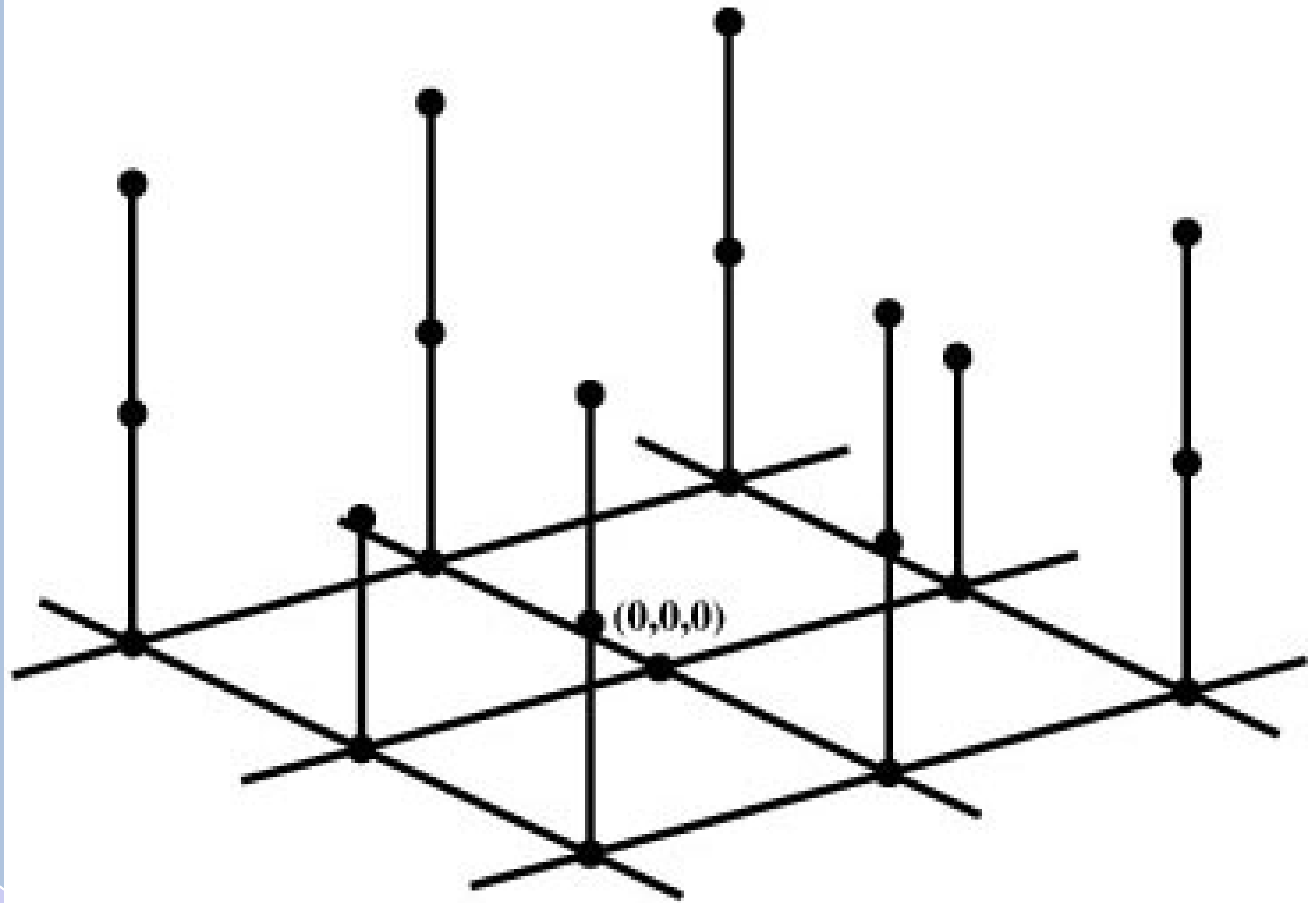
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Thank you!

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