Uniqueness and Extinction Properties of the Interacting Branching Collision Process

Progresses and Challenges of IBCP

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Outline

- Models (IBCP)

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- Two components: MBP and MCP: Revisited

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- Progress (II) for IBCP: Extinction Properties for Regular Case

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- Challenges from IBCP

Models

Def. 1 A conservative q-matrix $Q = \{q_{ij},\ i, j \in Z_+\}$ is called an Interacting Branching Collision q-matrix (IBC q-matrix) if it takes the form:

$$q_{ij} = \begin{cases} \frac{i(i-1)}{2} a_{j-i+2} + i b_{j-i+1} & \text{if } j \geq i-2,\ i \geq 2, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $a_j \geq 0 \quad (j \neq 2)$ and $-a_2 = \sum_{j \neq 2} a_j < +\infty$, together with $a_0 > 0$ and $\sum_{j=3}^{\infty} a_j > 0$. Also

$$b_j \geq 0 \quad (j \neq 1) \quad \text{and} \quad -b_1 = \sum_{j \neq 1} b_j < +\infty, \tag{2}$$

together with $b_0 > 0$, $b_{-1} = 0$ and $\sum_{j=2}^{\infty} b_j > 0$.

Models

Def. 2 An Interacting Branching Collision Process (IBCP) is a Z_+-valued CTMC whose transition function $P(t)$ satisfies the forward equation

$$P'(t) = P(t)Q \tag{3}$$

where Q is an IBC q-matrix.

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Def. 2 An Interacting Branching Collision Process (IBCP) is a Z_+-valued CTMC whose transition function $P(t)$ satisfies the forward equation

$$P'(t) = P(t)Q \tag{4}$$

where Q is an IBC q-matrix.

We see that

$$Q = Q^b + Q^c$$

where Q^b and Q^c are the conservative MBP and MCP q-matrices, respectively. The former process is well-known while the latter could be refereed to Chen et al JAP (2004).

Two components:

The first component is an MBP whose properties can be analysed by using the generating function of the sequence $\{b_j,\ j \geq 0\}$:

$$B(s) = \sum_{j=0}^{\infty} b_j s^j\,, \qquad |s| \leq 1.$$

Note that $B(0) = b_0 > 0$ and $B(1) = 0$.

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Note that $B(0) = b_0 > 0$ and $B(1) = 0$.

Also $B'(1) = \sum_{j=1}^{\infty} j b_{j+1} - b_0$ satisfies $-\infty < B'(1) \leq +\infty$.

Branching vs Collision: Revisited

Note also that the sign of $B'(1)$ determines the number of zeros of $B(s)$ in $[0,1]$.

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Lemma 1. The equation $B(s) = 0$ has at most two distinct roots in $[0, 1]$. More specifically, if $B'(1) \le 0$ then $B(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $B(s) = 0$ in $[0, 1]$, while if $B'(1) > 0$ (including $B'(1) = +\infty$) then $B(s) = 0$ has an additional root q_b satisfying $0 < q_b < 1$ such that $B(s) > 0$ for $0 \le s < q_b$ and $B(s) < 0$ for $q_b < s < 1$.

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Moreover, $B(s)=0$ does not have any other roots in the unit complex disk.

Regularity and Uniqueness

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Proposition 1. The MBP q-matrix Q^b is regular iff one of the following holds.
(i) $B'(1) < +\infty$.
(ii) $B'(1) = +\infty$ and

$$\int_{\varepsilon}^{1} \frac{1}{-B(s)} ds = +\infty$$

for some (or for all) $\varepsilon \in (q_b, 1)$, where $q_b < 1$ is the smallest nonnegative root of $B(s) = 0$, guaranteed by $B'(1) = +\infty$.

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Proposition 2. There always exists only one MBP which satisfies the Kolmogorov forward equations.

Extinction Probability

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Let $\{X(t),\ t \geq 0\}$ be the unique MBP and define the extinction times τ_0^b for states 0 by

$$\tau_0^b = \begin{cases} \inf\{t > 0,\ X(t) = 0\} & \text{if } X(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$q^{ib} = P\{\tau_0^b < +\infty | X(0) = i\}.$$

Branching vs Collision: Results

Proposition 3 The extinction probabilities of the MBP is given by

$$q^{ib} = q_b^i,$$

More specifically,

$$\begin{aligned} q^{ib} &= 1, && \text{if} \quad B'(1) \le 0, \\ q^{ib} &= q_b^i < 1, && \text{if} \; 0 < B'(1) \le +\infty. \end{aligned}$$

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In particular, the extinction probability is 1 for all $i > 0$ if and only if

$$B'(1) \leq 0$$

i.e. iff **the overall mean BIRTH rate $\leq$ the overall mean DEATH rate**.

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The second component is an MCP whose properties can be analysed by using the generating function of the sequence $\{a_j,\ j \geq 0\}$:

$$A(s) = \sum_{j=0}^{\infty} a_j s^j\,, \qquad |s| \leq 1.$$

This satisfies $A(0) = a_0 > 0$ and $A(1) = 0$.

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This satisfies $A(0) = a_0 > 0$ and $A(1) = 0$.

Also

$$A'(1) = \sum_{j=1}^{\infty} j a_{j+2} - 2a_0 - a_1$$

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The equation $A(s) = 0$ has a unique root η_c in $(-1, 0)$. Moreover, $A(s) = 0$ does not have any other roots in the unit complex disk.

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Extinction Probability

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Let $\{Y(t),\ t \geq 0\}$ be the unique MCP and define the extinction times τ_0^c and τ_1^c for states 0 and 1 by

$$\tau_0^c = \begin{cases} \inf\{t > 0,\ Y(t) = 0\} & \text{if } Y(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } Y(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

$$\tau_1^c = \begin{cases} \inf\{t > 0,\ Y(t) = 1\} & \text{if } Y(t) = 1 \text{ for some } t > 0 \\ +\infty & \text{if } Y(t) \neq 1 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$q^{i0} = P\{\tau_0^c < +\infty | Y(0) = i\} \quad \text{and} \quad q^{i1} = P\{\tau_1^c < +\infty | Y(0) = i\}.$$

Extinction Probability

Proposition 6 (i) If $A'(1) \leq 0$ then

$$\begin{aligned} q^{i0} &= (\eta_c^i - \eta_c)/(1 - \eta_c) \\ q^{i1} &= (1 - \eta_c^i)/(1 - \eta_c) \\ q^{i\infty} &= 0 \end{aligned}$$

(ii) If $0 < A'(1) \leq +\infty$ then

$$\begin{aligned} q^{i0} &= (q_c\eta_c^i - \eta_c q_c^i)/(q_c - \eta_c), \\ q^{i1} &= (q_c^i - \eta_c^i)/(q_c - \eta_c) \end{aligned}$$

and $q^{i\infty} = \left(q_c(1 - \eta_c^i) - \eta_c(1 - q_c^i) - (q_c^i - \eta_c^i)\right)/(q_c - \eta_c)$.

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where q_c is the smallest root of $A(s) = 0$ in $[0, 1]$
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In particular, the overall extinction probability $q^{i0} + q^{i1}$ is 1 if and only if

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Huge publications for MBP in literatures, see, in particular, T. E. Harris (1963), Athreya and Ney (1972), Asmussen and Hering(1983) and Athreya and Jagers (1996). For MCP, see A.V.Kalinkin (2002) and Chen, Pollett, Li and Zhang (2004, 2008).

IBCP: PDE

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Let $\{Z(t),\ t \geq 0\}$ be the unique IBCP and let

$$P(t) = \{p_{ij}(t)\}$$

and

$$R(\lambda) = \{r_{ij}(\lambda)\}$$

denote its transition function and resolvent, respectively.

IBCP: PDE

Theorem 3.1 (PDF) Suppose $P(t)$, $R(\lambda)$ are the Q-function and Q-resolvent of IBCP, respectively. Then

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{A(s)}{2}\frac{\partial^2 F_i(t,s)}{\partial s^2} + B(s)\frac{\partial F_i(t,s)}{\partial s}$$

and

$$\lambda G_i(\lambda,s) - s^i = \frac{A(s)}{2}\frac{\partial^2 G_i(\lambda,s)}{\partial s^2} + B(s)\frac{\partial G_i(\lambda,s)}{\partial s}$$

IBCP: PDE

where

$$F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j, \quad (i \geq 2),$$

and

$$G_i(\lambda,s) = \sum_{j=0}^{\infty} r_{ij}(\lambda)s^j, \quad (i \geq 2).$$

IBCP: Regularity

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Step (ii): "IF" part for case $0 < B'(1)$: Use the similar techniques as used in MBP.

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Step (i): "IF" part for case $B'(1) \leq 0$: Easy!

Step (ii): "IF" part for case $0 < B'(1)$: Use the similar techniques as used in MBP.

Step (iii): "ONLY IF" part: Use Comparison Technique (comparing with B-D-P) similarly as used in Chen et al JAP [2004].

IBCP: Uniqueness

Theorem 3.3. (**Uniqueness**) There always exists only one Q-function which satisfies the forward equations. That is that there always exists only one IBCP.

Progress on IBCP (II): Extinction

Let $\{Z(t),\ t \geq 0\}$ be the unique IBCP and define the extinction time τ by

$$\tau = \begin{cases} \inf\{t > 0,\ Z(t) = 0\} & \text{if } \ Z(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } \ Z(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$a_i = P\{\tau < +\infty | Z(0) = i\}$$

Progress on IBCP (II): Extinction

By the "experience" of MBP and MCP, it seems that we SHOULD have $a_i = 1$ iff

$$A'(1) + B'(1) \leq 0$$

i.e. $a_i = 1$ iff **overall mean BIRTH rate $\leq$ overall mean DEATH rate** .

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However, this guessing is INCORRECT, since the "contributions" made to the extinction by the two components **are NOT equivalent** ! Also, recall the two components **INTERACT** with each other! In fact, the extinction probabilities are much much more complicated than originally "expected"!!!

Extinction of IBCP: Regular Case

Recall IBCP is regular iff $A'(1) \leq 0$.

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Theorem 4.1 If $A'(1) \leq 0$ and $B'(1) \leq 0$, then

$$a_i \equiv 1 \quad (i \geq 1)$$

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Remaining case: $A'(1) = 0$ and $0 < B'(1) < +\infty$

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we need to introduce a "testing" function

$$H(y) = \exp\left\{2\int_0^y \frac{B(x)}{A(x)}dx\right\}$$

which possesses many interesting and important properties (but omitted here).

Extinction of IBCP: Regular Case

Now define

$$J = \int_{\eta_c}^{1} \frac{H(y)}{A(y)} dy$$

and

$$J_0 = \int_{0}^{1} \frac{H(y)}{A(y)} dy$$

then either $0 < J < +\infty$ or $J = +\infty$.
and $J = +\infty$ iff $J_0 = +\infty$

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Extinction of IBCP: Regular Case

Theorem 4.3 Suppose $A'(1) = 0$ and $0 < B'(1) < \infty$.

(i) If $J_0 = +\infty$, then

$$a_i = 1 \ \ (i \geq 1)$$

.

(ii) If $J_0 < \infty$ then

$$a_i = J^{-1} \cdot \int_{\eta_c}^{1} \frac{y^i H(y)}{A(y)} dy, \quad i \geq 1$$

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Theorem 4.4 Suppose $A'(1) = 0, 0 < B'(1) < +\infty$ and $A''(1) < \infty$.

(i) If $A''(1) \geq 4B'(1)$ then $J_0 = +\infty$ and thus

$$a_i = 1$$

.

(ii) If $A''(1) < 4B'(1)$ (including $B'(1) = +\infty$) then $J_0 < \infty$ and thus $a_i < 1$ and

$$a_i = J^{-1} \cdot \int_{\eta_c}^{1} \frac{y^i H(y)}{A(y)} dy, \quad i \geq 1$$

.

Extinction of IBCP: Irregular Case

Recall IBCP is irregular iff

$$A'(1) > 0$$

or, equivalently, iff

$$q_c < 1$$

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An irregular IBC q-matrix Q is called super-explosive if

$$q_b < q_c < 1$$

critical-explosive if

$$q_b = q_c < 1$$

or sub-explosive if

$$q_c < q_b \leq 1$$

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Theorem 5.1 If $q_b = q_c$, then $a_i = q_b^i$.

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Theorem 5.1 If $q_b = q_c$, then $a_i = q_b^i$.

Theorem 5.2 If Q is critical-explosive, then the mean conditional extinction time

$$E_i[\tau_0|\tau_0 < \infty]$$

is given by

$$E_i[\tau_0|\tau_0 < \infty] = q_c^{-i} \int_0^{q_c} [\frac{2}{H(s)} \int_{\eta_c}^{s} (1 - (\frac{y}{q_c})^i) \frac{H(y)}{A(y)} dy] ds$$

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Theorem 5.3 If $q_b < q_c < 1$ (super-explosive). Then the extinction probability a_i starting from $i \geq 1$, is

$$a_i = \frac{\int_{\eta_c}^{q_c} \frac{y^i H(y)}{A(y)} dy}{\int_{\eta_c}^{q_c} \frac{H(y)}{A(y)} dy}. \tag{6}$$

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Theorem 5.4 Suppose that $q_c < q_b \leq 1$ (sub-explosive). Further assume

$$A'(q_c) + 2B(q_c) = 0$$

Then

$$a_i = q_c^i + i\sigma q_c^{n-1} \tag{9}$$

where the positive constant σ is independent of i and given by

$$\sigma = -\frac{B(q_c)}{B'(q_c)}.$$

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$$a_i = q_c^i + i\sigma q_c^{n-1} \tag{10}$$

where the positive constant σ is independent of i and given by

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Extinction of IBCP: Irregular Case

Closed form could also be provided for another subcase of $A'(q_c) + 2B(q_c) < 0$

Extinction of IBCP: Irregular Case

Closed form could also be provided for another subcase of $A'(q_c) + 2B(q_c) < 0$

Theorem 5.5 Suppose the IBC q-matrix Q is sub-explosive and

$$A'(q_c) + 2B(q_c) < 0$$

Then

$$a_i = \frac{\int_{\eta_c}^{q_c} \frac{y^i B'(y) - i y^{i-1} B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy}{\int_{\xi_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy}.$$

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Definition of $A_1(x)$ and $B_1(x)$??

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We may get the following conclusion (details omitted including the definitions of $D_{m,k}$ etc.

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Remark: If

$$-2B(q_c)/A'(q_c)$$

is an integer, the problem is much simpler.

Challenges from IBCP:Open Qs

Still little is known about IBCP. Many important as well as interesting questions are hunting for their masters and homes. The following are some pets.

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Question 1 (**PDE**) More information from **PDE** (3.1) or **ODE** (3.2)?

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Question 1 (**PDE**) More information from **PDE** (3.1) or **ODE** (3.2)?

We are quite confident that the sequence of unknown functions $F_i(t, s)$ $(i \geq 1)$ can be expressed in terms of two independent functions, called $u(t, s)$ and $v(t, s)$, say. The questions are

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This may need hard but highly rewarding job and worth trying.

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Design and control of the interaction!

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Explosion probability and explosion time?

Distributions and conditional distributions of extinction time and explosion time? Also, other hitting times?

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More importantly and interestingly

Question 4 Decay parameter and QSD.

Until now we know nothing about them. Hence answer even some of the following is of significance .

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(iii) Effect of immigration and/or emigration on extinction probabilities, unconditional and/or conditional mean extinction times, explosions and QSD and conditional limiting distributions.

Challenges from MBCP:Open Qs

(iv) IBCP with continuous-state space.
Recall continuous-state space MBP for both jump and diffusion types.

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Question 6 **Applications**!

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Particularly in probability modelling of Biological, computing, and social (financial modelling, say) Sciences.

Acknowledgement

Thanks you all for your attention.