Exact Tail Asymptotics in a Generalized Markov Branching Process

Yuanyuan Liu

School of Mathematics, Central South University, China

liuyy@csu.edu.cn

Based on joint work with Prof. Yiqiang, Q. Zhao

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1. Introduction: GBMP Model

We consider a generalized Markov branching process (GMBP) with the infinitesimal generator $Q = (q_{ij})$ given by

$$q_{ij} = \begin{cases} h_j, & \text{if } i = 0, \\ i^\alpha p_{j-i+1}, & \text{if } i \geq 1 \text{ and } j \geq i-1, \\ 0, & \text{otherwise}, \end{cases} \qquad (1)$$

or in matrix form

$$Q = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & \cdots & \cdots \\ p_0 & p_1 & p_2 & p_3 & \cdots & \cdots \\ 0 & 2^\alpha p_0 & 2^\alpha p_1 & 2^\alpha p_2 & \cdots & \cdots \\ 0 & 0 & 3^\alpha p_0 & 3^\alpha p_1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

1. Introduction: GBMP Model

where $\alpha > 0$; $h_0 < 0$, $h_j \geq 0$ for $j \geq 1$; $p_1 < 0$, $p_j \geq 0$ for $j \neq 1$.

We assume that Q is irreducible and conservative and we make the convention that the GMBP is the minimal Q-process. The GMBP is called sublinear, linear or superlinear according to $\alpha < 1$, $\alpha = 1$ or $\alpha > 1$, respectively.

Studies on GMBP have been focusing on criteria for uniqueness and ergodicity, see for example, Chen, R.R. (1997), Zhang Y.H. (2001), Chen, A.Y. (2002), Lin, X., Zhang, H.J.(2006). Very little on performance properties is available, which are often key issues in applications.

1. Introduction: truncated Model

The truncated generator ${}_{(n)}Q$ of Q is given by

$$
{}_{(n)}Q = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-2} & \sum_{k=n-1}^{\infty} h_k \\ p_0 & p_1 & p_2 & \cdots & p_{n-2} & \sum_{k=n-1}^{\infty} p_k \\ 0 & 2^\alpha p_0 & 2^\alpha p_1 & \cdots & 2^\alpha p_{n-3} & \sum_{k=n-2}^{\infty} 2^\alpha p_k \\ 0 & 0 & 3^\alpha p_0 & \cdots & 3^\alpha p_{n-4} & \sum_{k=n-3}^{\infty} 3^\alpha p_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)^\alpha p_0 & -(n-1)^\alpha p_0 \end{pmatrix},
$$

which is obtained by augmenting the last column of the $n \times n$ north-west corner of Q into a (conservative) generator.

1. Introduction: truncated model

A truncated model arises if practically there is a limitation in population capacity, and also it is important if we use it as an approximation to the infinite-state branching process. In the literature, truncation approximations of stationary distribution are a well-known topic for discrete-time Markov chains (see e.g. Zhao,Y.Q. (1996), Tweedie, R.L. (1998)). However, little work is available for a continuous-time model.

1. Introduction: what to be studied

We are specifically interested in exact:

(a) tail asymptotics in the stationary distribution for a stable system;

(b) divergence rate in the partial sum of the (non-probability) invariant measure for a non-stable system;

(c) convergence rate to zero of the error in the stationary probability distributions between the truncated model and the corresponding original stable system;

(d) convergence rate to zero of the stationary probability for any fixed state for the truncated system obtained from a corresponding non-stable branching process.

2. Basics: lemma 1

Let ν be the invariant measure of Q (i.e. $\nu Q = 0$) and ${}_{(n)}\nu$ be the invariant measure of truncated matrix ${}_{(n)}Q$ (i.e., ${}_{(n)}\nu_{(n)}Q = 0$).

Lemma 1

$$ {}_{(n)}\nu_i = \frac{\nu_i}{S_\nu(n)}, \quad i = 0, 1, \ldots, n-1, \tag{2} $$

and if $\sum_{k=0}^{\infty} \nu_k = 1$, then

$$ \|{}_{(n)}\nu - \nu\|_1 := \sum_{k=0}^{n-1} \left| {}_{(n)}\nu_k - \nu_k \right| + \bar{S}_\nu(n) = 2\bar{S}_\nu(n), \tag{3} $$

where $S_x(n) = \sum_{j=0}^{n-1} x_j$ and $\bar{S}_x(n) = \sum_{j=n}^{\infty} x_j$ are defined for a sequence of real numbers x_n.

2. Basics: Remark 1

Remark 1

(i) (a) is closely related to (c)

According to (3), if the branching process is stable, then properties of asymptotics in the error are immediate results from the corresponding properties of asymptotics in the stationary distribution.

(ii) (b) is closely related to (d)

According to (2), if the branching process is unstable, then properties of asymptotics for ${}_{(n)}\nu_i$ for a fixed i are immediate consequences of that for the partial sum $S_\nu(n)$ of the invariant measure of Q.

2. Basics: Lemma 2

The study on asymptotics of the branching process is carried out by converting it into the study for the transition matrix of M/G/1 form as described below.

Lemma 2 ν is an invariant measure of Q if and only if π ($\pi_0 = \nu_0$ and $\pi_k = k^\alpha \nu_k, k \in \mathbb{N}$) is an invariant measure of the stochastic matrix P (referred to M/G/1 form) given by

$$P = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

2. Basics: Remark 2

where $b_0 = 1 + h_0$, $b_i = h_i$ for $i \neq 0$, $a_1 = 1 + p_1$ and $a_i = p_i$ for $i \neq 1$.

Remark 2 It should be noticed that P can be positive recurrent, null recurrent, or transient even if the GMBP is positive recurrent. Therefore, asymptotic analysis for invariant measures of P, not necessarily probability measure, is necessary for our study. The asymtotic analysis of invariant measures for non-stable P is not a well-addressed topic, which will be our focus.

Define $H(z)$ $P(z)$ $B(z)$ $A(z)$ and $\Pi(z)$ for h_k, p_k, b_k, a_k and π_k, respectively.

organiztion of the following parts

In the following, we will investigate exact asymptotics in ν through π for the three cases: $P'(1-) < 0$, $P'(1-) = 0$ and $0 < P'(1-) \leq \infty$, which are further divided into subcases according to properties of the boundary transitions.

Among all the cases, we exclude this case: $P'(1-) < 0$ and $H'(1-) < \infty$ in our study because the corresponding discrete-time P is positive recurrent, which is a classical topic in the literature studies, for example, readers may refer to Møller, J.R. (2001), Li Q. L. and Zhao, Y.Q. (2005) .

3. Negative drift: $P'(1-) < 0, H'(1-) = \infty$

To characterizatize the asymptotic property, we need the following assumption.

Assumption 1 Assume that $f(z)$ is a nonnegative function such that $f(z) \sim \frac{c}{(1-z)^\theta}$, as $z \to 1-$ for some $c > 0$ and $0 < \theta < 1$, where $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

A remark on the assumption will be given at the end of this section.

3. Theorem 1

Theorem 1 Suppose that $P'(1-) < 0$ and $H'(z)$ satisfies Assumption 1.
(i) If $\alpha < \theta$, then

$$S_\nu(n) \sim \frac{c\theta\nu_0}{(\theta-\alpha)(1-\theta)\Gamma(1+\theta)P'(1-)} n^{\theta-\alpha},$$

and for any fixed i

$${}_{(n)}\nu_i \sim \frac{(\theta-\alpha)(1-\theta)\Gamma(1+\theta)P'(1-)\nu_i}{c\theta\nu_0} n^{-(\theta-\alpha)}.$$

(ii) If $\alpha = \theta$, then

$$S_\nu(n) \sim \frac{c\theta\nu_0}{(1-\theta)\Gamma(1+\theta)P'(1-)} \log n,$$

3. Theorem 1

and for any fixed i

$$ {}_{(n)}\nu_i \sim \frac{(1-\theta)\Gamma(1+\theta)P'(1-)\nu_i}{c\theta\nu_0}(\log n)^{-1}. $$

(iii) If $\alpha > \theta$, then the branching process is positive recurrent,

$$ \bar{S}_\nu(n) \sim \frac{c\theta\nu_0}{(\alpha-\theta)(1-\theta)\Gamma(1+\theta)P'(1-)} n^{\theta-\alpha}, $$

and

$$ \|{}_{(n)}\nu - \nu\|_1 \sim \frac{2c\theta\nu_0}{(\alpha-\theta)(1-\theta)\Gamma(1+\theta)P'(1-)} n^{\theta-\alpha}. $$

3. Proof of Theorem 1

The proof of this theorem is based on the following lemmas.

lemma 3 Suppose that $A'(1-) < 1$. If the function $B'(z)$ satisfies Assumption 1, then

$$S_\pi(n) \sim \frac{c\pi_0}{(1-\theta)(1-A'(1-))\Gamma(1+\theta)} n^\theta.$$

Proof of this lemma needs to use Tauberian theorem.

3. Proof of Theorem 1

lemma 4 Suppose that $\sum_{k=1}^{n} k^{\alpha} x_k \sim n^{\gamma} \log^{\beta} n$ for the sequence of real numbers x_k, where $\gamma > 0$ and $\alpha, \beta \in \mathbb{R}$.

(i) If $\gamma > \alpha$, then $S_x(n) \sim \frac{\gamma}{\gamma-\alpha} n^{\gamma-\alpha} \log^{\beta} n$.

(ii) If $\gamma = \alpha$, then

$$S_x(n) \sim \frac{\gamma}{\beta+1} \log^{\beta+1} n, \quad \text{if } \beta > -1,$$

and

$$\overline{S}_x(n) \sim \frac{-\gamma}{\beta+1} \log^{\beta+1} n, \quad \text{if } \beta < -1.$$

(iii) If $\gamma < \alpha$, then the series $\sum_{k=1}^{\infty} x_k$ converges, and $\bar{S}_x(n) \sim \frac{\gamma}{\alpha-\gamma} n^{\gamma-\alpha} \log^{\beta} n$.

3. Remark on the assumption

Remark 3

(i) Assumption 1 covers many important cases, including the most commonly used distribution: zeta distribution $P[X=k] = \frac{k^{-\theta}}{\zeta(\theta)}$ when $1 < \theta < 2$, where $k \in \mathbb{N}_+$, X denotes an ordinary non-negative random variable and $\zeta(x)$ is the Riemann zeta function.

(ii) It is reasonable to restrict the value of θ in Assumption 1 to $\theta < 1$, since otherwise $\theta \geq 1$, from *Tauberian theorem*, the sereis $\sum_{k=0}^{\infty} b_k$ diverges.

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4. Zero drift: $P'(1-) = 0$

In the section, we divide our discussion into two subsections according to

$$H'(1-) < \infty \text{ or } H'(1-) = \infty$$

for the branching process. Each subsection is further divided into two parts depending on whether

$$P''(1-) < \infty \text{ or } P''(1-) = \infty.$$

The exact polynomial decay rates for all kinds of cases are shown in this section.

4.1.1. $H'(1-) < \infty, P''(1-) < \infty$

Theorem 2 Suppose that $P'(1-) = 0$, $H'(1-) < \infty$ and $P''(1-) < \infty$.

(i) If the branching process is sublinear (i.e. $\alpha < 1$), then

$$S_\nu(n) \sim \frac{H'(1-)\nu_0}{(1-\alpha)P''(1-)} n^{1-\alpha}.$$

(ii) If the branching process is linear (i.e., $\alpha = 1$), then

$$S_\nu(n) \sim \frac{H'(1-)\nu_0}{P''(1-)} \log n.$$

(iii) If the branching process is suplinear (i.e., $\alpha > 1$), then it is positive recurrent, and $\overline{S}_\nu(n) \sim \frac{H'(1-)\nu_0}{(\alpha-1)P''(1-)} n^{1-\alpha}$.

4.1.2. $H'(1-) < \infty, P''(1-) = \infty$

Theorem 3 Suppose that $P'(1-) = 0$, $H'(1-) < \infty$ and $P''(z)$ satisfies Assumption 1.

(i) If $\alpha < 1 - \theta$, then

$$S_\nu(n) \sim \frac{(2-\theta)(1-\theta)^2 H'(1-)\nu_0}{c(1-\theta-\alpha)\Gamma(2-\theta)} n^{1-\theta-\alpha}.$$

(ii) If $\alpha = 1 - \theta$, then

$$S_\nu(n) \sim \frac{(2-\theta)(1-\theta)^2 H'(1-)\nu_0}{c\Gamma(2-\theta)} \log n.$$

(iii) If $\alpha > 1 - \theta$, then the branching process is positive recurrent, and $\overline{S}_\nu(n) \sim \frac{(2-\theta)(1-\theta)^2 H'(1-)\nu_0}{c(\alpha-1+\theta)\Gamma(2-\theta)} n^{1-\theta-\alpha}$.

4.2.1. $H'(1-) = \infty, P''(1-) < \infty$.

Theorem 4 Suppose that $P'(1-) = 0$ and $P''(1-) < \infty$. Assume further that $H'(z)$ satisfies Assumption 1.

(1) If $\alpha < 1 + \theta$, then

$$S_\nu(n) \sim \frac{2c(1+\theta)\nu_0}{(1-\theta)(1+\theta-\alpha)\Gamma(2+\theta)P''(1-)} n^{1+\theta-\alpha}.$$

(2) If $\alpha = 1 + \theta$, then

$$S_\nu(n) \sim \frac{2c(1+\theta)\nu_0}{(1-\theta)\Gamma(2+\theta)P''(1-)} \log n.$$

(3) If $\alpha > 1 + \theta$, then the branching process is positive recurrent, and $\overline{S}_\nu(n) \sim \frac{2c(1+\theta)\nu_0}{(1-\theta)(\alpha-\theta-1)\Gamma(2+\theta)P''(1-)} n^{1+\theta-\alpha}$.

4.2.2. $H'(1-) = \infty, P''(1-) = \infty$.

Theorem 5 Suppose that $P'(1-) = 0$, $H'(z)$ satisfies Assumption 1 for constants $c > 0$ and $0 < \theta < 1$, and $P''(z)$ satisfies Assumption 1 for other constants $0 < \theta_1 < 1$ and $c_1 > 0$.

(i) If $\alpha < \theta + 1 - \theta_1$, then

$$S_\nu(n) \sim \frac{c(1-\theta_1)(2-\theta_1)(\theta+1-\theta_1)\nu_0}{c_1(1-\theta)(\theta+1-\theta_1-\alpha)\Gamma(2+\theta-\theta_1)} n^{\theta+1-\theta_1-\alpha},$$

(ii) If $\alpha = \theta + 1 - \theta_1$, then

$$S_\nu(n) \sim \frac{c(1-\theta_1)(2-\theta_1)(\theta+1-\theta_1)\nu_0}{c_1(1-\theta)\Gamma(2+\theta-\theta_1)} \log n,$$

4.2.2. $H'(1-) = \infty, P''(1-) = \infty$.

(iii) If $\alpha > \theta + 1 - \theta_1$, then the branching process is positive recurrent,

$$\overline{S}_\nu(n) \sim \frac{c(1-\theta_1)(2-\theta_1)(\theta+1-\theta_1)\nu_0}{c_1(1-\theta)(\alpha-\theta-1+\theta_1)\Gamma(2+\theta-\theta_1)} n^{\theta+1-\theta_1-\alpha},$$

5. Positive drift: $0 < P'(1-) \leq \infty$

We show below ν_n diverge to ∞ at a semi-geometric (an geometric function multiplied by a power function) rate, which is an interesting phenomenon.

Theorem 6 If $0 < P'(1-) \leq \infty$, then the branching process is not positive recurrent, and

$$\nu_n \sim -\frac{H(z_0)\nu_0}{P'(z_0-)} n^{-\alpha} z_0^{-n},$$

$$S_\nu(n) \sim -\frac{z_0 H(z_0)\nu_0}{(1-z_0)P'(z_0-)} n^{-\alpha} z_0^{-n},$$

where $z_0 \in (0,1)$ is the unique solution to the equation $P(z) = 0$.

*** The proof needs different argument.

Acknowledgement

Thank you !